

STABILITY FOR SEMILINEAR PARABOLIC PROBLEMS IN L_2 , $W^{1,2}$, AND INTERPOLATION SPACES

PAVEL GUREVICH AND MARTIN VÄTH

ABSTRACT. An asymptotic stability result for parabolic semilinear problems in $L_2(\Omega)$ and interpolation spaces is shown. Some known results about stability in $W^{1,2}(\Omega)$ are improved for semilinear parabolic mixed boundary value problems. The approach is based on Amann's power extrapolation scales. In a Hilbert space setting, a better understanding of this approach is provided for operators satisfying Kato's square root problem; as a side result some equivalent characterizations of these operators are obtained.

1. INTRODUCTION

To the authors knowledge, results dealing with linear stability of semilinear equations $u_t + Au = f(u)$ always make use of semigroup techniques. In the simplest of these results for C_0 -semigroups [25], the nonlinearity f is assumed to act (and be e.g. differentiable) in the same Banach space H in which the semigroup acts. In the case of heat equations or reaction-diffusion systems, i.e., when the semigroup is (essentially) given by the Laplace operator, the classical choices of the space H are e.g. $W^{1,p}(\Omega)$ (or subspaces taking some boundary conditions into account) or $L_p(\Omega)$. However, in these cases, the nonlinearity given by a superposition operator is differentiable if and only if it is affine, see e.g. [18].

One possible solution of this problem is to work in spaces of continuous functions, see [19]. However, this is not possible if one wants to consider Sobolev or L_p spaces. In this case, another approach can be found in [12], where the nonlinearity is assumed to act only from a space H_α with $\alpha \in [0, 1)$ into H with H_α being the domain of a fractional power of the (negative of the) generator of the semigroup. This idea can be extended to somewhat more general interpolation spaces, which in some cases avoids the problem that the space depends on the operator (which is important for quasilinear problems), see [5]. The classical folklore way to apply this result is to work in $H = L_p(\Omega)$, and one obtains that H_α is for sufficiently large p embedded into $C(\overline{\Omega})$, hence differentiability of the nonlinearity is not an issue anymore. However, one obtains asymptotic stability only in the space H_α with large $\alpha > 0$ since otherwise one ends up with very restrictive (or in case $\alpha = 0$ even degenerate) hypotheses about the nonlinearity f .

Results obtained in this way are usually not comparable with instability results for e.g. obstacle problems where one sometimes obtains instability in the $W^{1,2}(\Omega)$ or $L_2(\Omega)$ topology by completely different methods. In order to compare the problems with and without obstacles, we should thus know something about their linear stability in $W^{1,2}(\Omega)$ and $L_2(\Omega)$. Now the folklore way to do this is rather suboptimal. For a stability result for the Laplace

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operator with Neumann boundary conditions in the $W^{1,2}(\Omega)$ topology, we would need to consider $H = L_2(\Omega)$ and get $H_{1/2} = W^{1,2}(\Omega)$, hence our nonlinearity has to act from $W^{1,2}(\Omega)$ into $L_2(\Omega)$, which (in space dimension $N > 1$) amounts to a certain growth hypotheses on the function generating the superposition operator; a corresponding result for a reaction diffusion system was formulated e.g. in [29]. Moreover, to get a stability result in the $L_2(\Omega)$ -topology in this way, one would have to choose $\alpha = 0$, that is, one would need to consider the nonlinearity acting from H into itself. As mentioned above, this means that one cannot consider differentiable nonlinearities of superposition type at all.

Note that, in contrast, if one is interested in stationary solutions, i.e. in solutions of the corresponding elliptic problem, a natural approach is to consider the superposition operator acting from $W^{1,2}(\Omega)$ into the antidual space with respect to the L_2 -scalar product, i.e., into the antidual space $W^{1,2}(\Omega)'$. Since $L_p(\Omega) \subseteq W^{1,2}(\Omega)'$ for some $p < 2$, this approach requires a milder growth condition than if the nonlinearity acts from $W^{1,2}(\Omega)$ into $L_2(\Omega)$. It would be nice to have also a corresponding result with weaker growth hypothesis for the parabolic case.

It is perhaps not so well known that Amann's technique of power extrapolation spaces can be used to solve both problems simultaneously. One can obtain results about linear stability in $W^{1,2}(\Omega)$ under the "natural" acting conditions as in the elliptic problem (that is, for subcritical growth of the nonlinearity), thus relaxing the growth hypothesis supposed in e.g. [29]. Moreover, simultaneously, one can obtain stability result in the $L_2(\Omega)$ topology which is really applicable for superposition operators.

We note that stable manifolds using extrapolation spaces have also been introduced in [10] to obtain similarly a Hölder condition with respect to an averaging parameter. For particular parabolic equations similar approaches in $L_p(\Omega)$ with p close to 2 have been studied by K. Gröger, J. Rehberg, and others (see e.g. [11], particularly the proof of Lemma 5.3). The authors thank J. Rehberg for pointing out references to corresponding abstract results (personal communication).

The purpose of this paper is to carry out this technique, which is not straightforward, since e.g. spectral properties of perturbed operators do not carry over immediately to "extrapolated" operators. We begin with a Banach space setting and then concentrate on the case of an operator A generated by a "strongly accretive" form in a Hilbert space. For such an operator, one obtains an abstract extension \mathcal{A} in a natural manner. We will show that \mathcal{A} is generated by a "strongly accretive" form if and only if A solves Kato's square root problem. Moreover, this is the case if and only if \mathcal{A} is the "extrapolated" operator of A of order $-1/2$, and in this case all extrapolated/interpolated operators (of any negative or positive order) solve Kato's square root problem, too.

The plan of the paper is as follows. In Section 2, we recall (slight extensions of) the classical results related to stability from [12]. In Section 3, we extend these results under milder hypotheses about the nonlinearities in terms of Amann's extrapolated power scales. The rest of the paper is devoted to the Hilbert space setting, where the technique is particularly fruitful. In Section 4, we clarify the relation between these extrapolated power scales, strongly accretive operators, and Kato's square root problem. Applications to semilinear parabolic problems are given in Section 5; in particular, stability of a reaction-diffusion system is obtained, for which instability is known under obstacles [16]. In the appendix, we briefly discuss a sufficient condition for an operator to solve Kato's square root problem, which follows as a by-result of our main theorem of Section 4.

2. SUMMARY OF CLASSICAL RESULTS

Our main interest lies in some dynamical assertions about stability of equilibria for semi-linear parabolic equations, which we formulate now. We start by summarizing (slight extensions of) well-known results which can be found in e.g. [12].

Here and in the following, $(H, |\cdot|)$ denotes a complex Banach space, and $A: D(A) \rightarrow H$ a (densely defined closed) sectorial operator in H in the sense of [12], that is, $-A$ generates an analytic C_0 -semigroup. Moreover, we assume that the spectrum of A is disjoint from $(-\infty, 0]$. The latter implies that A is positive (of positive type) in the sense of [26] (or [5]), and it is actually no loss of generality, since it can be arranged by adding a corresponding multiple of the identity to A , if necessary.

Since A is of positive type, one can define fractional power operators A^α , $\alpha \in \mathbb{C}$. Here, we use Komatu's characterization of fractional power operators [17], which coincides with that of [26, Section 1.15.1] and that of [5]. For real $\alpha \geq 0$, we denote by H_α the domain of $D(A^\alpha) \subseteq H$, endowed with the norm

$$\|u\|_{H_\alpha} := |A^\alpha u|, \quad (2.1)$$

which is equivalent to the graph norm.

In this section, we fix $\alpha \in [0, 1]$; the case $\alpha = 0$, that is, $H_\alpha = H$ is explicitly admissible.

Given a subset $U \subseteq \mathbb{R} \times H_\alpha$ and a function $f: U \rightarrow 2^H$ (we include multi-valued f for completeness), we consider the problem

$$u'(t) + Au(t) \in f(t, u(t)). \quad (2.2)$$

Definition 2.1. We call $u \in C([t_0, t_1], H)$ a *strong/mild solution* of (2.2) if there is a function $f_0: (t_0, t_1) \rightarrow H$ with $f_0 \in L_1((t_0, \tau), H)$ for every $\tau < t_1$ such that the following holds for every $t \in (t_0, t_1)$: $(t, u(t)) \in U$, $f_0(t) \in f(t, u(t))$, and

(strong solution): $u'(t) \in H$ exists in the sense of the norm of H , $u(t) \in D(A)$, and $u'(t) + Au(t) = f_0(t)$.

(mild solution):

$$u(t) = e^{-(t-t_0)A}u(t_0) + \int_{t_0}^t e^{-(t-s)A}f_0(s)ds. \quad (2.3)$$

Theorem 2.2 (Classical Regularity). *Every strong solution is a mild solution, and the converse holds if f_0 in Definition 2.1 is locally Hölder continuous. Moreover, if $u: [t_0, t_1) \rightarrow H$ satisfies (2.3) for all $t \in (t_0, t_1)$ then*

- (1) *if $f_0 \in L_1((t_0, \tau), H)$ for every $\tau \in (t_0, t_1)$ then $u \in C([t_0, t_1), H)$.*
- (2) *if for every $\tau \in (t_0, t_1)$ there is $p > 1/(1 - \alpha)$ with $f_0 \in L_p((t_0, \tau), H_{-\gamma})$, then $u: (t_0, t_1) \rightarrow H_\alpha$ is locally Hölder continuous, and $u \in C([t_0, t_1), H_\alpha)$ if and only if $u(t_0) \in H_\alpha$.*

Proof. The first assertions can be found as e.g. [23, Corollary 4.2.2]. The remaining assertions follow by a standard calculation for weakly singular integrals (see e.g. [6, Satz 6.12] for the scalar case) by using that $e^{-tA}: H \rightarrow H_\alpha$ is bounded for $t > 0$ by C_0/t^α with C_0 independent of $t \geq 0$, that the function $g_{u_0}: [0, \infty) \rightarrow H_\alpha$, $g_{u_0}(t) := e^{-tA}u_0$ is locally Hölder continuous on $(0, \infty)$ if $u_0 \in H$ by [23, Theorem 2.6.3], and continuous at 0 if $u_0 \in H_\alpha$, because for $u_1 := A^\alpha u_0$ there holds $A^\alpha g(t) = e^{-tA}u_1$, see e.g. [23, Theorem 2.6.13(b,c)]. \square

Concerning existence results, we will for simplicity only consider single-valued f in which case we also get uniqueness and regularity. We say that f satisfies a *right local Hölder-Lipschitz* condition if for each $(t_0, u_0) \in U$ there is a (relative) neighborhood $U_0 \subseteq [t_0, \infty) \times$

H_α of (t_0, u_0) with $U_0 \subseteq U$ such that there are constants $L < \infty$ and $\sigma > 0$ with

$$|f(t, u) - f(s, v)| \leq L \cdot (|t - s|^\sigma + \|u - v\|_{H_\alpha}) \quad \text{for all } (t, u), (s, v) \in U_0. \quad (2.4)$$

We call f *left-locally bounded into H* if for each $t_1 > t_0$ and each bounded $M \subseteq H_\alpha$ there is some $\varepsilon > 0$ such that $f(U \cap ([t_1 - \varepsilon, t_1] \times M))$ is bounded in H .

Theorem 2.3 (Classical Uniqueness, Existence, Maximal Interval). (1) *If $f: U \rightarrow H$ satisfies a right local Hölder-Lipschitz condition, then for every $(t_0, u_0) \in U$ and $t_1 \in (t_0, \infty]$ there is at most one mild solution $u \in C([t_0, t_1], H_\alpha)$ of (2.2) satisfying $u(t_0) = u_0$.*
 (2) *Moreover, such a strong solution exists with some $t_1 > t_0$, and if f is left-locally bounded into H , then some maximal $t_1 > t_0$ can be chosen such that either $t_1 = \infty$ or $\|u(t)\|_{H_\alpha} \rightarrow \infty$ as $t \rightarrow t_1$ or the limit $u_1 = \lim_{t \rightarrow t_1^-} u(t)$ exists in H_α with $(t_1, u_1) \notin U$.*

Proof. The result is shown in the proofs of [12, Theorems 3.3.3 and 3.3.4]. We recall that local uniqueness implies global uniqueness by standard arguments. \square

Theorem 2.3 is only the motivation for the subsequent classical asymptotic stability result.

We formulate this result even for multi-valued $f: U \rightarrow 2^H$, since the proof is practically the same as in the classical single-valued case. We call $u_0 \in D(A)$ an *equilibrium* of (2.2) if $0 \in Au_0 + f(t, u_0)$ for all $t > 0$ and make the following hypothesis:

(B): Let u_0 be an equilibrium, $U_1 \subseteq H_\alpha$ an open neighborhood of u_0 and $[0, \infty) \times U_1 \subseteq U$. Assume that there is a bounded linear map $B: H_\alpha \rightarrow H$ such that the function $g(t, u) := f(t, u_0 + u) + Au_0 - Bu$ satisfies

$$\lim_{\|u\|_{H_\alpha} \rightarrow 0} \frac{\sup\{|v| : v \in g((0, \infty) \times \{u\})\}}{\|u\|_{H_\alpha}} = 0.$$

(Here, we use the convention $\sup \emptyset := 0$.)

If $f(t, \cdot)$ is single-valued in a neighborhood of u_0 , then $Au_0 = -f(t, u_0)$ so that hypothesis (B) means that $f(t, \cdot)$ is Fréchet differentiable at u_0 with derivative B , uniformly with respect to $t \in [0, \infty)$. We denote by $\sigma(A - B)$ the spectrum of $A - B$ in H .

Theorem 2.4 (Classical Asymptotic Stability). *Under hypothesis (B), assume that there is $\lambda_0 > 0$ such that $\sigma(A - B) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \lambda_0\}$.*

Then there exist $M_1, M_2 > 0$ such that if $t_1 > t_0 \geq 0$ and if $u \in C([t_0, t_1], H_\alpha)$ is a mild solution of (2.2) on $[t_0, t_1]$ with $\|u(t_0) - u_0\|_{H_\alpha} \leq M_1$, then u satisfies the asymptotic stability estimate

$$\|u(t) - u_0\|_{H_\alpha} \leq M_2 e^{-\lambda_0(t-t_0)} \|u(t_0) - u_0\|_{H_\alpha} \quad \text{for all } t \in [t_0, t_1]. \quad (2.5)$$

If f satisfies in addition the hypotheses of part (1) of Theorem 2.3, then additionally for every $t_0 \geq 0$ and every $u_1 \in H_\alpha$ with $\|u_1 - u_0\| \leq M_1$ a unique strong solution $u \in C([t_0, \infty), H_\alpha)$ with $u(t_0) = u_1$ exists and satisfies (2.5) with $t_1 = \infty$.

Proof. The result is proved analogously to [12, Theorem 5.1.1]. \square

The above classical results have several disadvantages. In the lack of a local Hölder-Lipschitz condition or, even more, in the multi-valued case, there may be solutions of (2.2) in a weaker sense which are not covered in Theorem 2.4. Moreover, in the most important case $H = L_2(\Omega)$ and when f is generated by a superposition operator, the choice $\alpha = 0$ is not possible, that is, one cannot obtain a nontrivial stability criterion in $H_0 = L_2(\Omega)$ by

Theorem 2.4. Indeed, it is well known that any differentiable (single-valued) superposition operator f in $L_2(\Omega)$ is actually affine, see e.g. [18].

In addition, even just the acting condition $f: U \rightarrow H$ in the spaces $H_\alpha = V = W^{1,2}(\Omega)$ and $H = L_2(\Omega)$ leads to a growth condition on f which appears unnecessarily restrictive. In the study of stationary solutions, one typically only requires that $f: V \rightarrow V'$ is continuous (and usually compact) which is satisfied under a much milder growth condition.

A solution of this problem is to replace the image space H in Theorems 2.3 and 2.4 by a larger space with a weaker topology. This can be done using Amann's extrapolated power scales.

3. RESULTS USING EXTRAPOLATED POWER SCALES

In this section, we make the same general hypotheses about A as in the previous section, that is, A is a densely defined sectorial operator with spectrum disjoint from $(-\infty, 0]$. We define the norm (2.1) on H also in case $\alpha < 0$. In general, H is not complete with respect to this norm, and so we define H_α for $\alpha < 0$ as the corresponding completion. With this notation, Amann's extrapolated power scale theory (see [4] or [5, Chapter V]) provides the following results.

All embeddings $H_\beta \subseteq H_\alpha$ with $\alpha < \beta$ are dense; they are all compact if and only if one of these embeddings is compact, and this is the case if and only if A has a compact resolvent.

For $\alpha \in \mathbb{R}$, A induces by graph closure (or restriction in case $\alpha \geq 0$) isomorphisms $A_\alpha: H_{1+\alpha} \rightarrow H_\alpha$ (hence A_α is closed as an operator in H_α by [5, Lemma I.1.1.2]). For $\beta > \alpha$, A_β is the H_β -realization of A_α , that is, $A_\beta = A_\alpha|_{D(A_\beta)}$ with $D(A_\beta) = A_\alpha^{-1}(H_\beta) = H_{\beta+1}$. All A_α are thus densely defined operators in H_α . They have the same spectrum as A and are sectorial in H_α (hence of positive type). In particular, $-A_\alpha$ generates an analytic semigroup in H_α . The corresponding semigroups correspond to each other by restriction or (unique) continuous extension, respectively.

It is remarkable that for the following result it is sufficient that A is a densely defined operator in a Banach space H of positive type. It follows by combining Proposition V.1.2.6 with Theorem V.1.3.9 (and their proofs) from [5], cf. e.g. [5, Corollary V.1.3.9].

Lemma 3.1. *There is a family of isometric isomorphisms $J_{\alpha,\beta}: H_\alpha \rightarrow H_\beta$ for $\alpha, \beta \in \mathbb{R}$ with $A_\alpha = J_{\alpha,\beta}^{-1} A_\beta J_{\alpha+1,\beta+1}$. In fact, $J_{\alpha,\beta} = (A_\alpha)^{\alpha-\beta}$ for $\alpha \leq \beta$, and $J_{\alpha,\beta} = J_{\beta,\alpha}^{-1} = A_\beta^{\alpha-\beta}$ for $\alpha \geq \beta$. Moreover, if $\gamma \geq 0$ then $J_{\alpha+\gamma,\beta+\gamma} = J_{\alpha,\beta}|_{H_{\alpha+\gamma}}$ is the $H_{\beta+\gamma}$ -realization of $J_{\alpha,\beta}$, that is, $H_{\alpha+\gamma} = J_{\alpha,\beta}^{-1}(H_{\beta+\gamma})$.*

Corollary 3.2. *Let $\alpha \in \mathbb{R}$, $\gamma \geq 0$. If $\sigma \leq 0$, then $A_{\alpha+\gamma}^\sigma = A_\alpha^\sigma|_{H_{\alpha+\gamma}}$. If $\sigma \geq 0$ then $A_{\alpha+\gamma}^\sigma = A_\alpha^\sigma|_{H_{\alpha+\gamma+\sigma}}$ is the $H_{\alpha+\gamma}$ -realization of A_α^σ , that is, $H_{\alpha+\gamma+\sigma} = (A_\alpha^\sigma)^{-1}(H_{\alpha+\gamma})$.*

We need to apply Amann's theory in different scales of spaces. The crucial observation for us is that there is a relation between these different scales. We already remarked that all our hypotheses which we assumed for (H, A) are also satisfied with the choice $(H_{-\gamma}, A_{-\gamma})$. Starting with this couple instead, we obtain by the above definition a corresponding family of spaces $(H_{-\gamma})_\alpha$. For instance, we have $(H_{-\gamma})_0 = H_{-\gamma}$. The following lemma states that these spaces are related to our original spaces H_α .

Lemma 3.3. *If $\alpha, \gamma \in \mathbb{R}$ then $H_\alpha = (H_{-\gamma})_{\alpha+\gamma}$.*

Proof. Set $\beta := \alpha + \gamma$. In case $\beta \geq 0$, we obtain from Lemma 3.1 that $A_{-\gamma}^\beta = J_{\alpha,-\gamma}$ is norm-preserving from H_α onto $H_{-\gamma}$. Hence, by the definition of $(H_{-\gamma})_\beta$, we obtain

$$u \in H_\alpha \iff A_{-\gamma}^\beta u \in H_{-\gamma} \iff u \in (H_{-\gamma})_\beta,$$

and the norm equality

$$\|u\|_{H_\alpha} = \|A_{-\gamma}^\beta u\|_{H_{-\gamma}} = \|u\|_{(H_{-\gamma})_\beta}.$$

In case $\beta \leq 0$, we obtain from Lemma 3.1 that $A_\alpha^\beta = J_{\alpha, -\gamma}$ is norm-preserving from H_α onto $H_{-\gamma}$. Using Corollary 3.2, we obtain

$$\|u\|_{H_\alpha} = \|A_\alpha^\beta u\|_{H_{-\gamma}} = \|A_{-\gamma}^\beta u\|_{H_{-\gamma}} = \|u\|_{(H_{-\gamma})_\beta}$$

for all $u \in H_{-\gamma}$. Since $H_{-\gamma}$ is densely embedded into H_α as well as into $(H_{-\gamma})_\beta$, the assertion follows. \square

Fixing now, throughout this section,

$$\alpha \in [0, 1), \quad \gamma \in [0, 1 - \alpha), \quad (3.1)$$

we relax the acting condition of f by replacing H by $H_{-\gamma}$ in the results of Section 2, that is, we require now only $f: U \rightarrow 2^{H_{-\gamma}}$ with $U \subseteq \mathbb{R} \times H_\alpha$.

Definition 3.4. We call $u \in C([t_0, t_1], H_{-\gamma})$ a γ -weak/mild solution of (2.2) if there is some $f_0: (t_0, t_1) \rightarrow H_{-\gamma}$ with $f_0 \in L_1((t_0, \tau), H_{-\gamma})$ for every $\tau \in (t_0, t_1)$ such that the following holds for every $t \in (t_0, t_1)$: $(t, u(t)) \in U$; $f_0(t) \in f(t, u(t))$, and

(γ -weak solution): $u'(t) \in H_{-\gamma}$ exists in the sense of the norm of $H_{-\gamma}$, $u(t) \in D(A_{-\gamma})$, and $u'(t) + A_{-\gamma}u(t) = f_0(t)$.

(γ -mild solution):

$$u(t) = e^{-(t-t_0)A_{-\gamma}}u(t_0) + \int_{t_0}^t e^{-(t-s)A_{-\gamma}}f_0(s)ds. \quad (3.2)$$

Remark 3.5. Since the semigroups are restrictions of each other, we can replace (3.2) equivalently by

$$u(t) = e^{-(t-t_0)A_{-\gamma_0}}u(t_0) + \int_{t_0}^t e^{-(t-s)A_{-\gamma_0}}f_0(s)ds$$

for every $\gamma_0 \geq \gamma$. We point this out, because in the subsequent Hilbert space setting, the operator $A_{-1/2}$ is “explicitly” given, and so it is natural to choose $\gamma_0 = 1/2$ in case $\gamma \leq 1/2$.

The purpose of relaxing the acting condition of f is that we can also relax the corresponding continuity hypotheses. We replace (2.4) by

$$\|f(t, u) - f(s, v)\|_{H_{-\gamma}} \leq L \cdot (|t - s|^\sigma + \|u - v\|_{H_\alpha}) \quad \text{for all } (t, u) \in U_0. \quad (3.3)$$

Similarly, we call f *left-locally bounded into* $H_{-\gamma}$ if for each $t_1 > t_0$ and each bounded $M \subseteq H_\alpha$ there is some $\varepsilon > 0$ such that $f(U \cap ([t_1 - \varepsilon, t_1] \times M))$ is bounded in $H_{-\gamma}$. Then we obtain the following generalization of Theorem 2.2.

Theorem 3.6 (Regularity). *Every γ -weak solution is a γ -mild solution, and the converse holds if f_0 in Definition 3.4 is locally Hölder continuous. Moreover, if $u: [t_0, t_1] \rightarrow H_{-\gamma}$ satisfies (3.2) for all $t \in (t_0, t_1)$ then*

- (1) *if $f_0 \in L_1((t_0, \tau), H_{-\gamma})$ for every $\tau \in (t_0, t_1)$ then $u \in C([t_0, t_1], H_{-\gamma})$.*
- (2) *if for every $\tau \in (t_0, t_1)$ there is $p > 1/(1 - \alpha)$ with $f_0 \in L_p((t_0, \tau), H_{-\gamma})$, then $u: (t_0, t_1) \rightarrow H_\alpha$ is locally Hölder continuous, and $u \in C([t_0, t_1], H_\alpha)$ if and only if $u(t_0) \in H_\alpha$.*

Proof. This is essentially Theorem 2.2 with (H, A, α) replaced by $(H_{-\gamma}, A_{-\gamma}, \beta)$ with $\beta := \alpha + \gamma$. Note that the semigroup generated by $A_{-\gamma}$ is indeed an extension of the semigroup generated by A . Moreover, by Lemma 3.3, the space $(H_{-\gamma})_\beta$ in the corresponding assertion of Theorem 2.2 is indeed the same as the space H_α in the assertion of Theorem 2.2. \square

In exactly the same way, the following result follows from Theorem 2.3.

Theorem 3.7 (Uniqueness, Existence, Maximal Interval). *Suppose (3.1).*

- (1) *If $f: U \rightarrow H_{-\gamma}$ satisfies a right local Hölder-Lipschitz condition in the sense (3.3), then for every $(t_0, u_0) \in U$ and $t_1 \in (t_0, \infty]$ there is at most one γ -mild solution $u \in C([t_0, t_1], H_\alpha)$ of (2.2) satisfying $u(t_0) = u_0$.*
- (2) *Moreover, a γ -weak solution exists with some $t_1 > t_0$, and if f is left-locally bounded into $H_{-\gamma}$, then some maximal $t_1 > t_0$ can be chosen such that either $t_1 = \infty$ or $\|u(t)\|_{H_\alpha} \rightarrow \infty$ as $t \rightarrow t_1$ or the limit $u_1 = \lim_{t \rightarrow t_1^-} u(t)$ exists in H_α with $(t_1, u_1) \notin U$.*

To generalize Theorem 2.4, we note that we assume now $f: U \rightarrow 2^{H_{-\gamma}}$ so that we have to generalize some notions.

Definition 3.8. An element $u_0 \in H_{1-\gamma}$ is called a γ -weak equilibrium of (2.2) if $A_{-\gamma}u_0 \in f(t, u_0)$ for every $t > 0$.

Since the operators are extensions of each other, we have:

Remark 3.9. If $0 \leq \tilde{\gamma} \leq \gamma$, then each $\tilde{\gamma}$ -weak equilibrium is a γ -weak equilibrium. Conversely, if u_0 is a γ -weak equilibrium with $A_{-\gamma}u_0 \in H_{-\tilde{\gamma}}$, that is, if $u_0 \in H_{1-\tilde{\gamma}}$, then u_0 is a $\tilde{\gamma}$ -weak equilibrium. Moreover, “0-weak equilibrium” means the same as “equilibrium”. In particular, each equilibrium u_0 is a γ -weak equilibrium, and the converse holds if $A_{-\gamma}u_0 \in H_{-0} = H$, that is, if $u_0 \in H_1 = D(A)$.

We will make the following hypothesis:

(B_γ): Let u_0 be a γ -weak equilibrium, $U_1 \subseteq H_\alpha$ an open neighborhood of u_0 , and $[0, \infty) \times U_1 \subseteq U$. Assume that there is a bounded linear map $B: H_\alpha \rightarrow H_{-\gamma}$ such that the function $g(t, u) := f(t, u_0 + u) + A_{-\gamma}u_0 - Bu$ satisfies

$$\lim_{\|u\|_{H_\alpha} \rightarrow 0} \frac{\sup\{\|v\|_{H_{-\gamma}} : v \in g((0, \infty) \times \{u\})\}}{\|u\|_{H_\alpha}} = 0.$$

Note that $H_{1-\gamma} \subseteq H_\alpha$, and so $B|_{H_{1-\gamma}}: H_{1-\gamma} \rightarrow H_{-\gamma}$ is bounded.

In the following result, we consider $A_{-\gamma} - B: H_{1-\gamma} \rightarrow H_{-\gamma}$ as an operator in $H_{-\gamma}$ with domain $H_{1-\gamma} \subseteq H_{-\gamma}$, and we denote the spectrum of this operator by $\sigma(A_{-\gamma} - B)$.

Theorem 3.10 (Asymptotic Stability). *Assume (3.1). Let hypothesis (B_γ) be satisfied. Suppose that there is $\lambda_0 > 0$ such that $\sigma(A_{-\gamma} - B) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \lambda_0\}$.*

Then there exist $M_1, M_2 > 0$ such that if $t_1 > t_0 \geq 0$ and $u \in C([t_0, t_1], H_\alpha)$ is a γ -mild solution of (2.2) with $\|u(t_0) - u_0\|_{H_\alpha} \leq M_1$, then u satisfies the asymptotic stability estimate (2.5).

If f satisfies in addition the hypotheses of part (1) of Theorem 3.7, then additionally for every $t_0 \geq 0$ and every $u_1 \in H_\alpha$ with $\|u_1 - u_0\| \leq M_1$ a unique γ -weak solution $u \in C([t_0, \infty), H_\alpha)$ with $u(t_0) = u_1$ exists and satisfies (2.5) with $t_1 = \infty$.

Proof. The result follows by applying Theorem 2.4 to the operator $A_{-\gamma}$ in the space $H_{-\gamma}$. \square

A stable manifold result in the spirit of Theorem 3.10 in, roughly speaking, the case $B = 0$ was shown in [10].

Theorem 3.10 is not as convenient as it appears at a first glance, because the operator $A_{-\gamma} - B$ is rather abstract, in general, and so its spectrum is hard to estimate. Therefore, we formulate two special cases in which this spectrum is “easier” to calculate.

The first case is described in the following result. Recall that (3.1) implies that

$$D(A) = H_1 \subseteq H_{1-\gamma} \subseteq H_\alpha = D(B) \subseteq H.$$

In particular, under the assumptions of the following result, $A - B$ is an operator in H with domain H_1 . Analogously to Theorem 2.4, we denote its spectrum by $\sigma(A - B)$.

Theorem 3.11. *Assume (3.1). Let hypothesis (\mathbf{B}_γ) be satisfied. Suppose that at least one of*

$$B(H_{1-\gamma}) \subseteq H \quad (3.4)$$

or

$$B(H_1) \subseteq H, \quad (3.5)$$

$$A_{-\gamma}u - Bu \in H \implies u \in H_1 \quad (3.6)$$

holds. Then $\sigma(A_{-\gamma} - B) = \sigma(A - B) \neq \mathbb{C}$. In particular, if $\lambda_0 > 0$ is such $\sigma(A - B) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \lambda_0\}$ then the conclusion of Theorem 3.10 holds with that λ_0 .

Proof. We first note that (3.4) implies (3.5) and (3.6), because $A: H_1 \rightarrow H$ is the H -realization of $A_{-\gamma}: H_{1-\gamma} \rightarrow H_{-\gamma}$. Moreover, (3.5) and (3.6) are equivalent to the assertion that $C_H := A - B: H_1 \rightarrow H$ is the H -realization of $C := A_{-\gamma} - B: H_{1-\gamma} \rightarrow H_{-\gamma}$.

Putting $\beta := \alpha + \gamma \in [0, 1)$, we have by Lemma 3.3 that $D(A_{-\gamma}^\beta) = (H_{-\gamma})_\beta = H_\alpha$. Since $B: H_\alpha \rightarrow H_{-\gamma}$ is bounded, and $A_{-\gamma}$ is sectorial, it follows that $A_{-\gamma} - B$ is sectorial, see e.g. [8, Remark 3.2].

Considering C as an operator in $H_{-\gamma}$ with domain $D(C) = H_{1-\gamma}$, we find in particular that there is $\mu > 0$ such that $\mu I + C$ has a bounded inverse R , and $R(H) \subseteq R(H_{-\gamma}) = D(C) \subseteq H$. Hence [5, Lemma V.1.1.1] implies that the spectra of C and of its H -realization C_H coincide. \square

The other special case of Theorem 3.10 concerns “weak” eigenvalues.

Definition 3.12. We call $\lambda \in \mathbb{C}$ a γ -weak eigenvalue of $A - B$ with eigenvector $u \in H_{1-\gamma}$, if λ is an eigenvalue of $A_{-\gamma} - B$ with eigenvector u .

Analogously to Remark 3.9, we obtain:

Remark 3.13. If $0 \leq \tilde{\gamma} \leq \gamma$ and λ is a $\tilde{\gamma}$ -weak eigenvalue of $A - B$, then λ is a γ -weak eigenvalue of $A - B$.

Conversely, if λ is a γ -weak eigenvalue of $A - B$ with eigenvector $u \in H_{1-\gamma} \subseteq H$ (recall that $\gamma \leq 1$) satisfying $Bu \in H_{-\tilde{\gamma}}$ or $u \in H_{1-\tilde{\gamma}}$, then λ is a $\tilde{\gamma}$ -weak eigenvalue of $A - B$ with eigenvector $u \in H_{1-\tilde{\gamma}}$.

Moreover, “0-weak eigenvalue” means the same as “eigenvalue”. In particular, each eigenvalue λ of $A - B$ is a γ -weak eigenvalue of $A - B$; conversely, if λ is a γ -weak eigenvalue of $A - B$ with eigenvector $u \in H_{1-\gamma}$ satisfying $Bu \in H$ or $u \in H_1$, then λ is an eigenvalue of $A - B$ with eigenvector $u \in H_1$.

Remark 3.13 implies in particular:

Proposition 3.14. *If at least one of (3.4) or (3.6) holds, then λ is a γ -weak eigenvalue of $A - B$ with eigenspace E if and only if λ is an eigenvalue of $A - B$ with the same eigenspace E , and automatically $E \subseteq D(A) = H_1$.*

Now we are in a position to formulate a variant of Theorem 3.10 in terms of eigenvalues instead of spectral values.

Theorem 3.15 (Asymptotic Stability with Eigenvalues). *Assume that one of the embeddings $H_\beta \rightarrow H_\delta$ is compact for $\beta > \delta$, that is, A has a compact resolvent. Suppose (3.1),*

and let hypothesis (\mathbf{B}_γ) be satisfied. Then $\sigma(A_{-\gamma} - B)$ consists only of the γ -weak eigenvalues of $A - B$, and the corresponding eigenspaces are finite-dimensional. More general, $A_{-\gamma} - B - \lambda I$ is a Fredholm operator of index 0 in $H_{-\gamma}$ for every $\lambda \in \mathbb{C}$.

In particular, if $\lambda_0 > 0$ is such that every γ -weak eigenvalue $\lambda \in \mathbb{C}$ of $A - B$ satisfies $\lambda > \lambda_0$, then the conclusion of Theorem 3.10 holds with that λ_0 .

Proof. Recall that the first hypothesis implies that all of the embeddings $H_\beta \rightarrow H_\delta$ are compact if $\beta > \delta$. In particular, the embedding $H_{1-\gamma} \rightarrow H_\alpha$ is compact.

Since $A_{-\gamma}: H_{1-\gamma} \rightarrow H_{-\gamma}$ is a Fredholm operator of index 0 (in the space $H_{-\gamma}$), it suffices to show by [15, Theorem 5.26] that $C := B + \lambda I: H_\alpha \rightarrow H_{-\gamma}$ is relatively compact with respect to $A_{-\gamma}$. Thus, let u_n and $A_{-\gamma}u_n$ be bounded in $H_{-\gamma}$. Then u_n is bounded in $H_{1-\gamma}$, and thus u_n contains a subsequence convergent in H_α . Hence, Cu_n contains a subsequence convergent in $H_{-\gamma}$, as required. \square

4. RELATIONS TO KATO'S SQUARE ROOT PROBLEM

For the rest of the paper, we pass to a Hilbert space setting. We assume that $(H, (\cdot, \cdot), |\cdot|)$ is a complex Hilbert space. We use the notation E' for the antidual space of a space E , and for an operator B in H , we denote by B^* the Hilbert space adjoint.

Let $(V, \|\cdot\|)$ be a complex Banach space which is densely embedded into H . The (Banach space) adjoint of the given embedding $i: V \rightarrow H$ defines the embedding $i': H' \rightarrow V'$ which has automatically a dense range, since i is one-to-one. Identifying H with H' and $i'(u)$ with an element of V' , we thus have a Gel'fand triple

$$V \subseteq H \subseteq V'. \quad (4.1)$$

As customary, we denote the pairing of V' and V also by (\cdot, \cdot) (which on $H \times V \subseteq H \times H$ coincides with the scalar product of H by definition of the adjoint, so that the notation is actually unique).

Throughout this section, let $a: V \times V \rightarrow \mathbb{C}$ be a sesquilinear form on V which is continuous, that is, there is $C \in [0, \infty)$ with

$$|a(u, v)| \leq C\|u\|\|v\|, \quad (4.2)$$

and which is strongly accretive in the sense that there is $c > 0$ with

$$\operatorname{Re} a(u, u) \geq c\|u\|^2 \quad \text{for all } u \in V. \quad (4.3)$$

The hypotheses (4.2) and (4.3) mean that $u \mapsto (\operatorname{Re} a(u, u) + |u|^2)^{1/2}$ defines an equivalent norm on V so that a is a closed form on $H \times H$ with domain $D(a) = V$ in the sense of [13, 15, 22].

Remark 4.1. For every $M \geq 0$, the sesquilinear form

$$b_M(u, v) := \frac{1}{2}(a(u, v) + \overline{a(v, u)} + M \cdot (u, v)) \quad (4.4)$$

is symmetric, that is, $b_M(v, u) = \overline{b_M(u, v)}$, and b_M satisfies estimates of the type (4.2) and (4.3). Hence, b_M becomes a scalar product on V , and the norm induced by this scalar product is equivalent to the norm on V . Thus, a form a satisfying (4.2) and (4.3) exists if and only if V is (isomorphic to) a Hilbert space.

We associate with a the linear operator $A: D(A) \rightarrow H$, defined by the duality $(Au, \cdot) = a(u, \cdot)$, that is, $D(A)$ is the set of all $u \in V$ for which there is some (uniquely determined) $Au \in H$ with

$$(Au, \varphi) = a(u, \varphi) \quad \text{for all } \varphi \in V.$$

Besides A , the form a also induces a bounded operator $\mathcal{A}: V \rightarrow V'$, defined by

$$(\mathcal{A}u, \varphi) = a(u, \varphi) \quad \text{for all } \varphi \in V, \quad (4.5)$$

where now the brace on the left denotes the antidual pairing.

The goal of this section is to answer the following questions (which will be made precise later on):

- (1) Is \mathcal{A} also associated to a strongly accretive form?
- (2) Is it true that \mathcal{A} corresponds to $A_{-1/2}$ from Section 3?

We will actually see that both answers are equivalent. They are equivalent to the assertion that A solves Kato's square root problem. Moreover, we will see that an analogous equivalence holds not only for $A_{-1/2}$ but actually for A_α with any $\alpha \in \mathbb{R}$.

We first summarize some well-known facts about A , see e.g. [15, Section VI] and [22, Proposition 1.51 and Theorem 1.52].

Proposition 4.2. *The operator A is closed and densely defined in H . The operators $A^{-1}: H \rightarrow H$ and $A^{-1}: V \rightarrow V$ are bounded, and $D(A)$ is dense in V . The operator $A: D(A) \rightarrow H$ is sectorial with spectrum contained in the open right half-plane, and $-A$ generates an analytic contraction C_0 -semigroup in H . If a is symmetric, then $A: D(A) \rightarrow H$ is selfadjoint in H .*

Analogous assertions hold also for \mathcal{A} , see e.g. [22, Theorem 1.55 and subsequent remarks].

Proposition 4.3. *The operator \mathcal{A} is an isomorphism of V onto V' . It is a densely defined sectorial operator in V' with spectrum in the open right half-plane.*

In view of Proposition 4.2, we are in the setting of Section 3, and so we can introduce the operators A^α , the spaces H_α , and $A_\alpha: H_{\alpha+1} \rightarrow H_\alpha$ as in that section. Moreover, also the adjoint operator A^* is of positive type, and $(A^\alpha)^* = (A^*)^\alpha$, see [5, Lemma V.1.4.11]. Let H_α^* denote the spaces of Section 3 generated by the operator A^* in place of A .

We consider also \mathcal{A} as an unbounded operator in V' with domain $D(\mathcal{A}) = V$. Also \mathcal{A} is of positive type, and thus \mathcal{A}^α are defined for all $\alpha \in \mathbb{C}$; for $\alpha \geq 0$, we endow $D(\mathcal{A}^\alpha)$ with the norms $\|u\|_{D(\mathcal{A}^\alpha)} = \|\mathcal{A}^\alpha u\|_{V'}$ for $\alpha \geq 0$ so that $\mathcal{A}^\alpha: D(\mathcal{A}) \rightarrow V'$ are isometric isometries.

Let us first note that the spaces H_α have a more convenient characterization in our setting. To this end, we denote by $[\cdot, \cdot]_\theta$ the complex interpolation functor of order θ , see e.g. [26].

Proposition 4.4. *For every $s \in \mathbb{R}$ the operator A^{is} is bounded in H with $\|A^{is}\| \leq e^{\pi|s|/2}$, and we have the reiteration formulas*

$$H_{(1-\theta)\alpha+\theta\beta} \cong [H_\alpha, H_\beta]_\theta \quad \text{if } \alpha, \beta \in \mathbb{R}, 0 < \theta < 1, \quad (4.6)$$

and the duality representation

$$H_{-\gamma} \cong (H_\gamma^*)' \quad \text{if } -1 \leq \gamma \leq 1. \quad (4.7)$$

Additionally,

$$H_\gamma^* \cong H_\gamma \quad \text{if } \gamma \in [0, 1/2). \quad (4.8)$$

If a is symmetric, then $H_{1/2}^* = H_{1/2} \cong V$.

Proof. The norm estimate $\|A^{is}\| \leq e^{\pi|s|/2}$ is shown in [14, Theorem 5]. Using this, we obtain the reiteration formula (4.6) from [5, Theorem V.1.5.4]. The duality formula (4.7) can be found as [5, Proposition V.1.5.5] (note [5, Remark V.1.5.16]). The assertion (4.8) follows from [13, Theorem 1.1]. For symmetric a , we have $A = A^*$, hence $H_{1/2} = H_{1/2}^*$, and the assertion $H_{1/2} \cong V$ is shown in [13] (see also [22, Theorem 8.1]). \square

The last assertion of Proposition 4.4 suggests the following definition.

Definition 4.5. We call an operator A in H a *Kato operator* (with a form on V) if it is the operator associated with a sesquilinear form $a: V \times V \rightarrow \mathbb{C}$ satisfying (4.2) and (4.3) and $H_{1/2} \cong V$.

Recall that we assumed throughout that A is associated with a form a on V satisfying (4.2) and (4.3). Thus, A is a Kato operator if and only if $H_{1/2} \cong V$.

Proposition 4.6. *A is a Kato operator if and only if $H_{1/2}^* \cong H_{1/2}$. In particular, A is a Kato operator if and only if*

$$H_\gamma^* \cong H_\gamma \quad \text{for all } \gamma \in [0, 1/2]. \quad (4.9)$$

A is a Kato operator if and only if A^ is a Kato operator.*

Proof. The first assertion is a special case of [14, Theorem 1], and (4.9) follows in view of (4.8). The last assertion follows from the previous assertion and the observation that A^* is generated by the form $a^*(u, v) := \overline{a(v, u)}$, see e.g. [22, Proposition 1.24]. \square

The name in Definition 4.5 is motivated by Kato's famous square root problem originally posed in [13]: to characterize the forms a for which A is a Kato operator. According to Proposition 4.4, A is a Kato operator if a is symmetric. However, also many elliptic differential operators (even nonsymmetric) induce Kato operators, see e.g. [22, Chapter 8] and the references therein as well as [3, 9, 24]. So the requirement that A is a Kato operator is rather mild from the viewpoint of applications we have in mind.

The reason why Kato operators are so useful to us in the Hilbert space setting is the following. In applications to partial differential equations the form a and the space V are usually explicitly given, while the spaces H_α and the operators A_α are known only implicitly; in fact, usually even $D(A) = H_1$ is known only implicitly. For Kato operators, we understand these auxiliary spaces and operators in case $\alpha \geq -1/2$ much better as the following simple observation shows.

Proposition 4.7. *If (and only if) A is a Kato operator, we have*

$$V \cong H_{1/2}, \quad V' \cong (H_{1/2}^*)' \cong H_{1/2}' \cong H_{-1/2}, \quad A_{-1/2} = \mathcal{A}.$$

Moreover, in this case, if $\gamma \in (0, 1/2)$, then

$$H_{-\gamma} \cong (H_\gamma^*)' \cong H_\gamma' \cong [H, V]_{2\gamma}' \cong [V', V]_{\frac{1}{2}+\gamma}' \cong [V', V]_{\frac{1}{2}-\gamma} \cong [V', H]_{1-2\gamma}, \quad (4.10)$$

$H = H_0 \cong [V', V]_{1/2}$, and if $\gamma \in (1/2, 1)$ then

$$H_{-\gamma} \cong (H_\gamma^*)' \cong [D(A^*)', V']_{2-2\gamma} \cong [V, D(A^*)]_{2\gamma-1}'. \quad (4.11)$$

Independently of whether A is a Kato operator, there holds

$$H_{-\gamma} \cong (H_\gamma^*)' \cong [H, D(A^*)']_\gamma \cong [H, D(A^*)]_\gamma' \quad \text{for all } \gamma \in (0, 1). \quad (4.12)$$

Proof. The first assertion is the definition of a Kato operator, the second follows by using (4.9) and (4.7). The identity $A_{-1/2} = \mathcal{A}$ (which has to be interpreted in terms of the canonical isomorphisms, of course), follows from the density of $D(A)$ in V (Proposition 4.2), since both operators are bounded from V into V' and coincide on $D(A)$ with A . The formula (4.12) is shown in a straightforward manner with (4.6) and (4.7) by inserting $D(A^*) = H_1^*$ and $H = H_0 = H_0' = H_0^* = (H_0^*)'$. The formulas (4.10), $H \cong [V', V]_{1/2}$, and (4.11) are shown similarly, by using also (4.9) and $V \cong H_{1/2} \cong H_{1/2}^*$. \square

Corollary 4.8. *The last three equality signs in (4.10) and $[V', V]_{1/2} \cong H$ are valid even if A fails to be a Kato operator.*

Proof. The claimed equalities are actually independent of A and a . Hence, in view of Remark 4.1, we can assume that a is symmetric, and in this case Proposition 4.4 implies that the associated self-adjoint operator A is a Kato operator, and the equalities follow from Proposition 4.7. \square

Remark 4.9. Let $\gamma \in [0, 1/2]$ and A be a Kato operator. Then λ is a γ -weak eigenvalue of $A - B$ with corresponding eigenvector $u \neq 0$ if and only if $u \in H_{1-\gamma}$ and

$$a(u, \varphi) - (Bu, \varphi) = \lambda(u, \varphi) \quad \text{for all } \varphi \in V.$$

Moreover, the hypothesis of Proposition 3.14 is satisfied if $B(V) \subseteq H$.

Indeed, the first assertion follows from the fact that $A_{-\gamma}$ is a restriction of $A_{-1/2} = \mathcal{A}$. The second assertion follows from $H_{1-\gamma} \subseteq H_{1/2} \cong V$.

Now we consider the question whether the operator \mathcal{A} is associated with a strongly accretive continuous sesquilinear form \mathfrak{a} on H . We first have to equip V' with an appropriate scalar product. Our idea for this is to fix a scalar product b on V which generates a norm $\|u\|_b := \sqrt{b(u, u)}$ on V equivalent to $\|\cdot\|$. Note that $\mathcal{A}: V \rightarrow V'$ is an isomorphism, and so $\|u\|_{a,b} := \|\mathcal{A}^{-1}u\|_b$ defines an equivalent norm in V' . We denote by $X^{a,b}$ the Hilbert space which we obtain from V' when we pass to this equivalent norm induced by the scalar product

$$[u, v]_{a,b} := b(\mathcal{A}^{-1}u, \mathcal{A}^{-1}v) \quad \text{for all } u, v \in X^{a,b}. \quad (4.13)$$

Note that for any choice of (a, b) as above we have $X^{a,b} \cong V'$. However, it is crucial for our approach to distinguish the various scalar products.

We emphasize that (4.13) is actually the general form of a scalar product on V' generating an equivalent norm. Indeed, if $[\cdot, \cdot]_{V'}^*$ is any scalar product on V' such that the generated norm $\|u\|_{V'}^* := \sqrt{[u, u]_{V'}^*}$ is equivalent to $\|\cdot\|_{V'}$, then $\|u\|_V^* := \|\mathcal{A}u\|_{V'}^*$ is a norm on V which is equivalent to $\|\cdot\|$. Moreover, this norm is generated by the scalar product $b(u, v) := [\mathcal{A}u, \mathcal{A}v]_{V'}^*$, and the corresponding scalar product (4.13) is just the scalar product $[\cdot, \cdot]_{V'}^*$ we started with.

The following result characterizes those scalar products b on V (or, equivalently, those scalar products on V') generating an equivalent norm for which a “strongly accretive” continuous form \mathfrak{a} on H exists such that \mathcal{A} is associated to \mathfrak{a} .

Proposition 4.10. *The following assertions are equivalent for every $c_1 > 0$.*

- (1) *There exists a sesquilinear form $\mathfrak{a}: H \times H \rightarrow \mathbb{C}$ such that there are constants $c_2, c_3 \geq 0$ with*

$$\operatorname{Re} \mathfrak{a}(u, u) \geq c_1 |u|^2, \quad |\mathfrak{a}(u, v)| \leq c_2 |u| |v|, \quad |\mathfrak{a}(u, u)| \leq c_3 |u|^2 \quad (4.14)$$

for all $u, v \in H$, and

$$\mathfrak{a}(u, v) = [\mathcal{A}u, v]_{a,b} \quad \text{for all } u \in V, v \in H. \quad (4.15)$$

- (2) *There is $c_2 \geq 0$ with*

$$\operatorname{Re} b(u, A^{-1}u) \geq c_1 |u|^2 \quad \text{and} \quad |b(u, A^{-1}v)| \leq c_2 |u| |v| \quad \text{for all } u, v \in D(A). \quad (4.16)$$

- (3) *There is $c_3 \geq 0$ with*

$$\operatorname{Re} b(u, A^{-1}u) \geq c_1 |u|^2 \quad \text{and} \quad |b(u, A^{-1}u)| \leq c_3 |u|^2 \quad \text{for all } u \in D(A). \quad (4.17)$$

One can equivalently replace $D(A)$ by V in (2) and (3), and the smallest possible constants c_2, c_3 in the above assertions are respectively the same. If (1) holds, then \mathbf{a} is the unique continuous function $\mathbf{a}: H \times H \rightarrow \mathbb{C}$ satisfying (4.15), and \mathcal{A} is the operator associate to \mathbf{a} in the Hilbert space $X^{a,b}$, that is, $u \in H$ belongs to $D(\mathcal{A}) \cong V$ if and only if there is some $w \in X^{a,b}$ with $\mathbf{a}(u, \varphi) = [w, \varphi]_{a,b}$ for all $\varphi \in H$.

Proof. “(1) \implies (2) \implies (3) \implies (2)”: Let $\mathbf{a}_1: V \times V \rightarrow \mathbb{C}$ be the sesquilinear form defined by $\mathbf{a}_1(u, v) = [\mathcal{A}u, v]_{a,b}$. Due to (4.15), we have $\mathbf{a}|_{V \times V} = \mathbf{a}_1$. Since V is dense in H , \mathbf{a} is uniquely determined by its restriction to $V \times V$, and moreover, there is at most one continuous function $\mathbf{a}: H \times H \rightarrow \mathbb{C}$ with $\mathbf{a}|_{V \times V} = \mathbf{a}_1$.

The operator L associated with the form \mathbf{a} is the Friedrichs extension of \mathcal{A} . Since \mathcal{A} is sectorial in the sense of [12], it follows from [15, Theorem VI.2.9] that $L = \mathcal{A}$.

Using (4.13), the fact that A is the H -realization of \mathcal{A} , and (4.5), we calculate

$$\mathbf{a}_1(u, v) = b(u, \mathcal{A}^{-1}v) = b(u, A^{-1}v) \quad \text{for all } u, v \in V. \quad (4.18)$$

Hence, (4.14) implies (4.16) and (4.17) even for all $u, v \in V$ (with the same constants c_2, c_3). Clearly, (4.16) implies (4.17) with $c_3 := c_2$. Conversely, if (4.17) holds, then $|\mathbf{a}_1(u, u)| \leq c_3|u|^2$ for all $u \in D(A)$, and an application of the polarization identity [15, (VI.1.1)] for the sesquilinear form \mathbf{a}_1 shows that \mathbf{a}_1 is bounded with respect to the norm $|\cdot|$. This means that (4.16) holds with some $c_2 \geq 0$.

“(2) \implies (1)”: If (4.16) holds then, since $D(A)$ is dense in V and since the left-hand side is continuous in V , we obtain that (4.16) holds even for all $u, v \in V$. (With the same argument also (4.17) holds for all $u, v \in V$.) Hence, by (4.18),

$$\operatorname{Re} \mathbf{a}_1(u, u) \geq c_1|u|^2 \quad \text{and} \quad |\mathbf{a}_1(u, v)| \leq c_2|u||v| \quad \text{for all } u, v \in V.$$

Since V is dense in H , it follows that the sesquilinear form \mathbf{a}_1 has a continuous extension $\mathbf{a}: H \times H \rightarrow \mathbb{C}$, and \mathbf{a} satisfies (4.14) (with the same constants $c_1, c_2 > 0$). \square

Proposition 4.10 motivates the following definition.

Definition 4.11. We call a scalar product b on V an *A-Kato scalar product* if the norm induced by b is equivalent to $\|\cdot\|$, and if (4.16) or, equivalently, (4.17) (or any of these with $D(A)$ replaced by V) are satisfied with some $c_1 > 0$ and $c_2, c_3 \geq 0$.

We will see that the existence of an *A-Kato scalar product* b on V is actually equivalent to the assertion that A is a Kato operator with a form on V , but at the moment we cannot use this.

Corollary 4.12. *If a is symmetric, then $b := a$ is an A-Kato scalar product on V for which \mathbf{a} in Proposition 4.10 is given by $\mathbf{a}(u, v) = (u, v)$ in H .*

Proof. It suffices to prove that assertion (1) of Proposition 4.10 holds. We calculate for \mathbf{a}_1 from the proof of Proposition 4.10 for all $u, v \in V$, using (4.13) and (4.5), that

$$\overline{\mathbf{a}_1(u, v)} = [v, \mathcal{A}u]_{a,a} = a(\mathcal{A}^{-1}v, \mathcal{A}^{-1}\mathcal{A}u) = (\mathcal{A}\mathcal{A}^{-1}v, u) = \overline{(u, v)}.$$

Hence, $\mathbf{a}_1(u, v) = (u, v)$ for all $u, v \in V$, and by continuity and density \mathbf{a}_1 has a unique continuous extension $\mathbf{a}(u, v) = (u, v)$ for all $u, v \in H$. \square

Remark 4.13. Corollary 4.12 may appear rather surprising. The form \mathbf{a} generating \mathcal{A} (and thus also defining A) is actually independent of a (and thus of A)! However, the explanation for this apparent contradiction is that the scalar product in $X^{a,a} \cong V'$ heavily depends on a , of course. Although the generated norms are equivalent, they are not the same, in general.

Note that the definition of an A -Kato scalar product does not make use of any sesquilinear form. Hence, it is remarkable that the following result makes sense and holds for every densely defined operator A of positive type in a Banach space H .

Lemma 4.14. *For $\alpha, \beta \in \mathbb{R}$, let $J = J_{\beta+1/2, \alpha+1/2}: H_{\beta+1/2} \rightarrow H_{\alpha+1/2}$ be the isometric isomorphism of Lemma 3.1. If b is an A_α -Kato scalar product on $H_{\alpha+1/2}$, then*

$$B(u, v) := b(Ju, Jv)$$

defines an A_β -Kato scalar product on $H_{\beta+1/2}$ with the same constants in the respective inequalities of (4.16) and (4.17) and with the same equivalence constants for the norm.

Proof. By Lemma 3.1, J is the $H_{\alpha+1/2}$ -realization of the isometric isomorphism $J_{\beta, \alpha}: H_\beta \rightarrow H_\alpha$, $J_{\beta+1, \alpha+1}$ is a restriction of J , and $A_\beta = J_{\beta, \alpha}^{-1} A_\alpha J_{\beta+1, \alpha+1}$. Hence, $JA_\beta^{-1} = J_{\beta+1, \alpha+1} A_\beta^{-1} = A_\alpha^{-1} J_{\beta, \alpha}$. We obtain

$$B(u, A_\beta^{-1}v) = b(Ju, A_\alpha^{-1}(Jv)) \quad \text{and} \quad \|Ju\|_{H_{\alpha+1/2}} = \|u\|_{H_{\beta+1/2}}$$

for all $u, v \in H_{\beta+1/2}$ from which the assertion follows straightforwardly. \square

Lemma 4.15. *Let $\alpha, \beta \in \mathbb{R}$ and $J = J_{\beta, \alpha}: H_\beta \rightarrow H_\alpha$ denote the isometric isomorphism of Lemma 3.1. Let b_α be a scalar product on H_α generating an equivalent norm and a_α a sesquilinear form on $H_{\alpha+1/2}$ satisfying*

$$|a_\alpha(u, v)| \leq C\|u\|_{H_{\alpha+1/2}}\|v\|_{H_{\alpha+1/2}}, \quad \operatorname{Re} a_\alpha(u, u) \geq c\|u\|_{H_{\alpha+1/2}}^2 \quad \text{for all } u, v \in H_{\alpha+1/2}.$$

Then

$$b_\beta(u, v) := b_\alpha(Ju, Jv), \quad a_\beta(u, v) := a_\alpha(Ju, Jv)$$

define a scalar product on H_β generating an equivalent norm with the same equivalence constants as for b_α and a sesquilinear form on $H_{\beta+1/2}$, respectively, such that

$$|a_\beta(u, v)| \leq C\|u\|_{H_{\beta+1/2}}\|v\|_{H_{\beta+1/2}}, \quad \operatorname{Re} a_\beta(u, u) \geq c\|u\|_{H_{\beta+1/2}}^2 \quad \text{for all } u, v \in H_{\beta+1/2}.$$

Moreover, if A_α is the operator associated to a_α with the scalar product b_α , then A_β is the operator associated to a_β with the scalar product b_β .

Proof. Note that $J_{1/2} := J_{\beta+1/2, \alpha+1/2}: H_{\beta+1/2} \rightarrow H_{\alpha+1/2}$ and $J_1 := J_{\beta+1, \alpha+1}: H_{\beta+1} \rightarrow H_{\alpha+1}$ are norm preserving isomorphisms and the corresponding $H_{\alpha+1/2}$ -realization and $H_{\alpha+1}$ -realization of J , respectively, and $A_\beta = J^{-1} A_\alpha J_1$. In particular,

$$\begin{aligned} D(A_\beta) &= H_{\beta+1} = J^{-1}(H_{\alpha+1}) = J^{-1}(D(A_\alpha)), \quad JA_\beta u = A_\alpha Ju \quad (u \in D(A_\beta)) \\ H_{\beta+1/2} &= J^{-1}(H_{\alpha+1/2}), \quad \|Ju\|_{H_{\alpha+1/2}} = \|u\|_{H_{\beta+1/2}} \quad (u \in H_{\beta+1/2}). \end{aligned} \tag{4.19}$$

Hence, $b_\beta(A_\beta u, v) = b_\alpha(A_\alpha Ju, Jv) = a_\alpha(Ju, Jv) = a_\beta(u, v)$ for all $u \in D(A_\beta)$, $v \in H_{\beta+1/2}$. Using this and (4.19), the assertions follow straightforwardly. \square

Now we can formulate the main result of this section. It states that if A is a Kato operator, then also each of the operators A_α is a Kato operator if H_α is equipped with an appropriate scalar product. As a by-result, we obtain some equivalent characterizations of Kato operators, in particular the previously mentioned observation that Kato operators are characterized by the existence of an A -Kato scalar product.

Theorem 4.16. *Assume the general hypotheses of this section, that is, A is associated to a sesquilinear form a on V satisfying (4.2) and (4.3). Then the following assertions are equivalent:*

- (1) *A is a Kato operator with a form on V , that is, $H_{1/2} \cong V$.*
- (2) *$A^{1/2}$ is an isomorphism of V onto H .*

- (3) *There is an A -Kato scalar product on V .*
- (4) *There is a scalar product on V' generating an equivalent norm and a sesquilinear form $\mathbf{a}: H \times H \rightarrow \mathbb{C}$ satisfying (4.14) with $c_1 > 0$ and $c_2, c_3 \geq 0$ such that \mathcal{A} is associated to \mathbf{a} .*
- (5) *There are $\varepsilon > 0$ and $M \geq 0$ such that for all $s \in (-\varepsilon, \varepsilon)$ the operator \mathcal{A}^{is} is bounded in V' with $\|\mathcal{A}^{is}\| \leq M$.*
- (6) *$D(\mathcal{A}^{1/2}) \cong H$.*
- (7) *$\mathcal{A} = A_{-1/2}$.*
- (8) *There is a scalar product on V' generating an equivalent norm such that \mathcal{A} is a Kato operator with a form on H .*
- (9) *For some $\alpha \in \mathbb{R}$ the space H_α can be equipped with a scalar product generating an equivalent norm such that the operator A_α is a Kato operator with a form on $H_{\alpha+1/2}$.*
- (10) *For every $\alpha \in \mathbb{R}$ the space H_α can be equipped with a scalar product generating an equivalent norm such that the operator A_α is a Kato operator with a form on $H_{\alpha+1/2}$.*

In each case, $b(u, v) := (A^{1/2}u, A^{1/2}v)$ defines an A -Kato scalar product on V , and there is $M > 0$ such that for all $s \in \mathbb{R}$ the operator \mathcal{A}^{is} is bounded in V' with $\|\mathcal{A}^{is}\| \leq Me^{\pi|s|/2}$. One can even choose $M = 1$ if one equips V' with the equivalent norm from (4).

Proof. “(1) \iff (2)” follows from the fact that $A^{1/2}$ is an isomorphism of $H_{1/2}$ onto $H_0 = H$.

“(1),(2) \implies (3)”: We show that $b(u, v) := (A^{1/2}u, A^{1/2}v)$ is an A -Kato scalar product on V . By hypothesis, there are constants $C_1, C_2 > 0$ with

$$C_1|A^{1/2}v| \leq \|v\| \leq C_2|A^{1/2}v| \quad \text{for all } v \in V \cong H_{1/2}.$$

Hence, in this case we calculate for every $u \in V$, noting that $u \in H_{1/2}$ and thus $v := A^{-1/2}u \in H_1$, that

$$b(u, A^{-1}u) = (A^{1/2}u, A^{1/2}A^{-1}u) = (Av, v) = a(v, v)$$

and $|A^{1/2}v| = |u|$. This implies on the one hand that

$$\operatorname{Re} b(u, A^{-1}u) \geq c\|v\| \geq C_1c|A^{1/2}v| = C_1c|u|,$$

and on the other hand

$$|b(u, A^{-1}u)| \leq C\|v\|^2 \leq C_2^2C|A^{1/2}v| = C_2^2C|u|^2.$$

Hence, (4.17) holds with $c_1 := C_1c$ and $c_3 := C_2^2C$.

“(3) \iff (4)” is the content of Proposition 4.10 and its preceding remarks.

“(4) \implies (5)”: Equipping the space V' with the scalar product and corresponding norm from (4), we obtain even the uniform estimate $\|\mathcal{A}^{is}\| \leq e^{\pi|s|/2}$ for all $s \in \mathbb{R}$ by applying Proposition 4.4 with (A, a, H) replaced by $(\mathcal{A}, \mathbf{a}, V')$.

“(5) \implies (6)”: Applying [26, Theorem 1.15.3] with the operator \mathcal{A} in V' , we find $D(\mathcal{A}^{1/2}) \cong [D(\mathcal{A}^0), D(\mathcal{A}^1)]_{1/2} = [V', V]_{1/2}$. Hence, (6) follows from Corollary 4.8.

“(6) \implies (7)” was shown in the proof of Proposition 4.7.

“(7) \implies (1)” follows from the calculation $H_{1/2} = D(A_{-1/2}) = D(\mathcal{A}) \cong V$.

“(4),(6) \iff (8)” is our definition of a Kato operator.

So far, we have shown the equivalences of all assertions of Theorem 4.16 with the exception of (9) and (10). Since “(10) \implies (1)” and “(1) \implies (9)” are obvious, we now have to show “(9) \implies (10)”. Thus, let α denote the number for which the assertion (9) holds, and let $\beta \in \mathbb{R}$ be arbitrary. By our choice of α , the general hypotheses of our section hold with (H_α, A_α) in place of (H, A) , and by Lemma 4.15, the same is true for (H_β, A_β) . Hence, using the already shown implications “(1) \implies (3)” with (H_α, A_α) and “(3) \implies (1)” with (H_β, A_β) , respectively, we obtain in view of Lemma 4.14 that A_β is a Kato operator. \square

Some of the direct equivalences of Theorem 4.16 are contained implicitly in [7, Section 5.5.2]. For instance, the equivalent characterization (6) of Kato operators was obtained from these results and applied in [3]. However, it seems that the idea to employ appropriate scalar products on V appears to be new.

5. APPLICATIONS TO SEMILINEAR PARABOLIC PDES

5.1. Superposition Operators in L_2 and Sobolev Spaces. Let $\Omega \subseteq \mathbb{R}^d$ be open and $H := L_2(\Omega, \mathbb{C}^n)$. In the following, we use the scalar product (and respective dual pairing)

$$(u, v) := \int_{\Omega} u(x) \cdot \overline{v(x)} dx.$$

In case $d \geq 3$, we put $p_* := \frac{2d}{d-2}$; in case $d \leq 2$, we fix an arbitrary $p_* \in (2, \infty)$. Let $V \subseteq W^{1,2}(\Omega, \mathbb{C}^n)$ be a closed subspace which is dense in H . We assume that Ω is such that Sobolev's embedding theorem is valid in the sense that there is a continuous embedding $V \subseteq L_{p_*}(\Omega, \mathbb{C}^n)$.

Remark 5.1. For the case that Ω is such that the dense embedding $V \subseteq L_{p_*}(\Omega, \mathbb{C}^n)$ holds only with some smaller power $p_* \in (2, \infty)$, all subsequent considerations hold as well with this choice of p_* .

Lemma 5.2. *Let A be a Kato operator.*

(1) *Let $\gamma \in [0, 1/2]$.*

(a) *We have a continuous embedding $L_{q_\gamma}(\Omega, \mathbb{C}^n) \subseteq H'_\gamma \cong (H^*_\gamma)' \cong H_{-\gamma}$ with*

$$q_\gamma := \left(\frac{1}{2} + \gamma - \frac{2\gamma}{p_*} \right)^{-1} \quad \left(= \frac{2d}{d+4\gamma} \in \left[\frac{2d}{d+2}, 2 \right] \text{ if } d \geq 3 \right). \quad (5.1)$$

(b) *If we have a continuous embedding $D(A) \subseteq L_p(\Omega, \mathbb{C}^n)$ ($1 \leq p < \infty$), then we also have a continuous embedding $H_{1-\gamma} \subseteq L_{p_\gamma}(\Omega, \mathbb{C})$ with*

$$p_\gamma := \left(\frac{2\gamma}{p_*} + \frac{1-2\gamma}{p} \right)^{-1}. \quad (5.2)$$

(2) *Let $\gamma \in [1/2, 1]$.*

(a) *We have a continuous embedding $H_{1-\gamma} \subseteq L_{p_\gamma}(\Omega, \mathbb{C})$ with*

$$p_\gamma := \left(\gamma - \frac{1}{2} + \frac{2-2\gamma}{p_*} \right)^{-1} \quad \left(= \frac{2d}{4\gamma+d-2} \in \left[\frac{2d}{d+2}, 2 \right] \text{ if } d \geq 3 \right). \quad (5.3)$$

(b) *If we have a continuous embedding $D(A^*) \subseteq L_p(\Omega, \mathbb{C}^n)$ ($1 \leq p < \infty$), then we also have a continuous embedding $L_{q_\gamma}(\Omega, \mathbb{C}^n) \subseteq (H^*_\gamma)' \cong H_{-\gamma}$ with*

$$q_\gamma := \left(1 - \frac{2\gamma-1}{p} - \frac{2-2\gamma}{p_*} \right)^{-1}. \quad (5.4)$$

Proof. By hypothesis, we have a continuous dense embedding $V \subseteq L_{p_*}(\Omega, \mathbb{C}^n)$. Hence, with $\frac{1}{p'_*} + \frac{1}{p_*} = 1$ also the (Banach space) adjoint embedding $L_{p'_*}(\Omega, \mathbb{C}^n) \subseteq V'$ is continuous and dense. In case $\gamma = 1/2$ we have $q_\gamma = p_\gamma = p'_*$, and thus the assertion (2) follows. In case $\gamma = 0$ we have $q_\gamma = 2$ and $p_\gamma = p$, and the assertion (1) is trivial. In case $\gamma \in (0, 1/2)$, we use [26, Theorem 1.18.4], the fact that $[\cdot, \cdot]_\theta$ is an interpolation functor of order θ (see e.g. [26, Theorem 1.9.3(a)]), and (4.10). Then we have a continuous embedding

$$L_{q_\gamma}(\Omega, \mathbb{C}^n) \cong [L_{p'_*}(\Omega, \mathbb{C}^n), L_2(\Omega, \mathbb{C}^n)]_{1-2\gamma} \subseteq [V', H]_{1-2\gamma} \cong H'_\gamma,$$

which proves (1a).

Defining p' by $\frac{1}{p'} + \frac{1}{p} = 1$, we find in view of $H_1^* = D(A^*) \subseteq L_p(\Omega, \mathbb{C}^n)$ that $L_p'(\Omega, \mathbb{C}^n) = L_p(\Omega, \mathbb{C}^n)' \subseteq (H_1^*)'$. This shows (2) for $\gamma = 1$, since $q_\gamma = p'$ and $p_\gamma = 2$. Moreover, for $\gamma \in (1/2, 1)$, we find similarly as above with (4.11) the continuous embedding

$$L_{q_\gamma}(\Omega, \mathbb{C}^n) = [L_{p'}(\Omega), L_{p_*}(\Omega, \mathbb{C}^n)]_{2-2\gamma} \subseteq [D(A^*)', V']_{2-2\gamma} \cong (H_\gamma^*)',$$

which implies (2b). A similar argument shows with Proposition 4.4 that in case $\gamma \in (0, 1/2)$

$$H_{1-\gamma} \cong [H_{1/2}, H_1]_{1-2\gamma} \subseteq [L_{p_*}(\Omega, \mathbb{C}^n), L_p(\Omega, \mathbb{C}^n)]_{1-2\gamma} \cong L_{p_\gamma}(\Omega, \mathbb{C}^n),$$

proving (1b), while in case $\gamma \in (1/2, 1)$

$$H_{1-\gamma} \cong [H_0, H_{1/2}]_{2-2\gamma} \subseteq [L_2(\Omega, \mathbb{C}^n), L_{p_*}(\Omega, \mathbb{C}^n)]_{2-2\gamma} \cong L_{p_\gamma}(\Omega, \mathbb{C}^n),$$

proving (2a) (all embeddings being continuous). \square

We assume also that the nonlinearity $f(t, \cdot)$ is given by a superposition operator induced by a function $\tilde{f}: [0, \infty) \times \Omega \times \mathbb{C}^n \rightarrow 2^{\mathbb{C}^n}$, that is, for each $t \in [0, \infty)$

$$f(t, u) := \{v: \Omega \rightarrow \mathbb{C}^n \mid v \text{ measurable and } v(x) \in \tilde{f}(t, x, u(x)) \text{ for almost all } x \in \Omega\}. \quad (5.5)$$

For the stability result, without loss of generality, we will consider only the case $u_0 = 0$ and assume that \tilde{f} is uniformly linearizable at $u = 0$ in the following sense. There are $r \in (1, \infty]$, a measurable $\tilde{B}: \Omega \rightarrow \mathbb{C}^{n \times n}$, and a function $\tilde{g}: (0, \infty) \times \Omega \times \mathbb{C}^n \rightarrow 2^{\mathbb{C}^n}$ with

$$\tilde{f}(t, x, u) = \tilde{B}(x)u + \tilde{g}(t, x, u) \quad \text{for all } (t, x, u) \in (0, \infty) \times \Omega \times \mathbb{C}^n$$

such that

$$\lim_{|u| \rightarrow 0} \frac{\sup\{|v| : v \in \tilde{g}((0, \infty) \times \{x\} \times \{u\})\}}{|u|} = 0 \quad (5.6)$$

for almost all $x \in \Omega$. Moreover, we assume that there is $C_0 \in (0, \infty)$ such that

$$\sup\{|v| : v \in \tilde{g}((0, \infty) \times \{x\} \times \{u\})\} \leq C_0 \cdot (|u| + |u|^\sigma) \quad \text{for all } u \in \mathbb{C}^n \quad (5.7)$$

for almost all $x \in \Omega$ and some $\sigma \in (1, \infty)$. We define a corresponding multiplication operator B by

$$Bu(x) := \tilde{B}(x)u(x) \quad \text{for all } x \in \Omega. \quad (5.8)$$

With this notation, the following holds.

Proposition 5.3. *Let A be a Kato operator and $u_0 = 0$. Suppose that*

$$\Omega \text{ has finite (Lebesgue) measure} \quad (5.9)$$

and that $r \in [1, \infty]$ and $\sigma \in (0, \infty)$ are such that $\tilde{B} \in L_r(\Omega, \mathbb{C}^{n \times n})$ and (5.6) and (5.7) hold.

(1) *Let $\alpha = 0$. Assume*

$$\begin{cases} r = 2 & \text{if } d = 1, \\ r > 2 & \text{if } d = 2, \\ r = \frac{2p_*}{p_* - 2} \quad (= d) & \text{if } d \geq 3, \end{cases} \quad (5.10)$$

and

$$\begin{cases} \sigma = 2 & \text{if } d = 1, \\ \sigma < 2 & \text{if } d = 2, \\ \sigma = 2 - \frac{2}{p_*} \quad \left(= 1 + \frac{2}{d} \right) & \text{if } d \geq 3. \end{cases} \quad (5.11)$$

Then for every $\gamma \in [1/2, 1)$ there holds $f: [0, \infty) \times H_\alpha \rightarrow 2^{(H_\gamma^)'}$, and the hypothesis (\mathbf{B}_γ) of Theorems 3.10 and 3.15 is satisfied with $H_\alpha = L_2(\Omega, \mathbb{C}^n)$.*

- (2) Let $\alpha = 0$. Assume that the embedding $D(A^*) \subseteq L_p(\Omega, \mathbb{C}^n)$ is continuous for some $p \in (p_*, \infty)$, and

$$r > \frac{2p}{p-2} \quad \text{and} \quad \sigma < 2 - \frac{2}{p}. \quad (5.12)$$

Then

$$\gamma_0 := \frac{(\sigma - 2)p_*p - 2p_* + 4p}{4(p - p_*)} < 1, \quad \gamma_1 := \frac{2pp_* - r(pp_* + 2p_* - 4p)}{4(p - p_*)r} < 1, \quad (5.13)$$

and for every $\gamma \in [\max\{\gamma_0, \gamma_1, 1/2\}, 1)$ the same conclusion as in (1) is valid.

- (3) Let $\alpha = 1/2$. Suppose that $\tilde{B} \in L_r(\Omega, \mathbb{C}^{n \times n})$ with some

$$r > \frac{p_*}{p_* - 2} \quad \left(= \frac{d}{2} \text{ if } d \geq 3 \right), \quad (5.14)$$

and that (5.6) and (5.7) hold with some

$$\sigma < p_* - 1 \quad \left(= \frac{d+2}{d-2} \text{ if } d \geq 3 \right). \quad (5.15)$$

Then

$$\gamma_0 := \frac{2\sigma - p_*}{2p_* - 4} < \frac{1}{2} \quad \text{and} \quad \gamma_1 := \frac{p_*}{r(p_* - 2)} - \frac{1}{2} < \frac{1}{2}, \quad (5.16)$$

and for every $\gamma \in [\max\{0, \gamma_0, \gamma_1\}, 1/2)$ we have $f: [0, \infty) \times H_\alpha \rightarrow 2^{H'_\gamma}$, and the hypothesis (\mathbf{B}_γ) of Theorems 3.10 and 3.15 is satisfied with $H_\alpha = V$.

Proof. In case (1), it is no loss of generality to assume $\gamma = 1/2$, and we assume first $d \geq 3$. In cases (1) and (2), we put $\tilde{p} = 2$ and define q_γ by (5.4), while in case (3), we put $\tilde{p} = p_*$ and define q_γ by (5.1). Then we put $U := L_{\tilde{p}}(\Omega, \mathbb{C}^n)$ and $V_\gamma := L_{q_\gamma}(\Omega, \mathbb{C}^n)$. Letting r satisfy (5.10), (5.12), or (5.14), and requiring $\gamma \geq \gamma_1$ with γ_1 as in (5.13) or (5.16) in the respective cases, we find

$$\frac{1}{q_\gamma} \geq \frac{1}{\tilde{p}} + \frac{1}{r},$$

and so we obtain from the (generalized) Hölder inequality that $B: U \rightarrow V_\gamma$ is bounded. Since we have a bounded embedding $H_\alpha \subseteq U$, we obtain from Lemma 5.2 that $B: H_\alpha \rightarrow (H_\gamma^*)'$ is bounded.

Moreover, letting σ satisfy (5.11), (5.12), or (5.15), and requiring $\gamma \geq \gamma_0$ with γ_0 as in (5.13) or (5.16) in the respective cases, we find $\sigma \leq \tilde{p}/q_\gamma$. Hence, the superposition operator g generated by \tilde{g} satisfies $g: [0, \infty) \times U \rightarrow 2^{V_\gamma}$ and

$$\lim_{\|u\|_U \rightarrow 0} \frac{\sup\{\|v\|_{V_\gamma} : v \in g((0, \infty) \times \{u\})\}}{\|u\|_U} = 0,$$

see [28, Theorem 4.14]. Since we have continuous embeddings $H_\alpha \subseteq U$ and $V_\gamma \subseteq (H_\gamma^*)'$ (Lemma 5.2), the condition (\mathbf{B}_γ) is proved.

Case (1) with $d = 2$ is treated in a similar way (with a sufficiently large p_*), and for $d = 1$ we can put $q_\gamma = 1$ in the above calculation, since in this case we have still a continuous embedding $V_\gamma \subseteq (H_\gamma^*)'$ by the continuity of the embedding $(H_\gamma^*)' \subseteq H_{1/2} \subseteq L_\infty(\Omega, \mathbb{C}^n)$. \square

Remark 5.4. The last observation in the proof extends to a more general situation: If $\gamma \in [0, 1/2]$ is such that the embedding $H_\gamma \subseteq L_\infty(\Omega, \mathbb{C}^n)$ is continuous, then the conclusion of Proposition 5.3(1) is valid with $r = \sigma = 2$ (we put $q_\gamma = 1$ in the proof).

Remark 5.5. For $d \geq 3$ assertion (2) of Proposition 5.3 requires strictly less about r and σ than assertion (1), because in view of $p > p_*$ there holds

$$\frac{2p}{p-2} < \frac{2p_*}{p_*-2} \quad \text{and} \quad 2 - \frac{2}{p} > 2 - \frac{2}{p_*}.$$

Remark 5.6. In case $d \geq 3$ the quantities γ_0 and γ_1 in (5.16) have the form

$$\gamma_0 = \frac{(d-2)\sigma - d}{4} \quad \text{and} \quad \gamma_1 = \frac{1}{2} \left(\frac{d}{r} - 1 \right). \quad (5.17)$$

Remark 5.7. Proposition 5.3(3) holds also with $\gamma = 1/2$. Moreover, for $\gamma = 1/2$ one does not have to require that the inequalities in (5.14) or (5.15) are strict. However, the choice $\gamma = 1/2$ violates the hypothesis (3.1) of Theorems 3.10 and 3.15 if $\alpha = 1/2$.

Remark 5.8. Hypothesis (5.9) is obviously needed for the assertion (3) of Proposition 5.3. However, we used this hypothesis also for the assertion (1) when we applied [28, Theorem 4.14]. If hypothesis (5.9) fails, one can apply other criteria for the differentiability of superposition operators like e.g. [28, Theorem 4.9], but we do not formulate corresponding results here.

While Proposition 5.3 gives a sufficient condition for the hypothesis (\mathbf{B}_γ) , this is not sufficient to apply Theorem 3.15. For the latter, one also has to estimate all γ -weak eigenvalues of $A - B$, and the latter in turn is usually simpler if one knows that all γ -weak eigenvalues of $A - B$ are eigenvalues of $A - B$. For the operator B from (5.8), this is the content of the following result.

Proposition 5.9. *Suppose (5.9). Let B have the form (5.8) with some $\tilde{B} \in L_r(\Omega, \mathbb{C}^n)$, $r \in [1, \infty]$.*

- (1) *If r satisfies (5.10), then $B|_V: V \rightarrow H$ is bounded.*
- (2) *If A is a Kato operator, $\gamma \in [1/2, 1)$, and*

$$\gamma \leq \tilde{\gamma}_0 := \begin{cases} 1 - \frac{p_*}{(p_*-2)r} & \text{if } r < \infty, \\ 1 & \text{if } r = \infty, \end{cases} \quad (5.18)$$

then $B|_{H_{1-\gamma}}: H_{1-\gamma} \rightarrow H$.

- (3) *If A is a Kato operator,*

$$2 < r < \frac{2p_*}{p_*-2} \quad (= d \text{ if } d \geq 3), \quad (5.19)$$

and if there is $p \geq \frac{2r}{r-2}$ ($> p_$) with $D(A) \subseteq L_p(\Omega, \mathbb{C}^n)$, then*

$$\tilde{\gamma}_p := \frac{1}{2} \left(\frac{1}{2} - \frac{1}{r} - \frac{1}{p} \right) \cdot \left(\frac{1}{p_*} - \frac{1}{p} \right)^{-1} \in [0, 1/2), \quad (5.20)$$

and for all $\gamma \leq \tilde{\gamma}_p$ the operator $B|_{H_{1-\gamma}}: H_{1-\gamma} \rightarrow H$ is bounded.

If A is a Kato operator, the hypotheses of either (1), (2), or (3) are satisfied and $\gamma \leq 1/2$, $\gamma \leq \tilde{\gamma}_0$, or $\gamma \leq \tilde{\gamma}_p$, respectively, then λ is a γ -weak eigenvalue of $A - B$ if and only if λ is an eigenvalue of $A - B$.

Proof. In case (1) with $d \geq 3$, we apply in view of

$$\frac{1}{2} = \frac{1}{p_*} + \frac{1}{r}$$

the (generalized) Hölder inequality to obtain that $B: L_{p_*}(\Omega, \mathbb{C}^n) \rightarrow L_2(\Omega, \mathbb{C}^n)$ is bounded and thus $B: V \rightarrow H$ is bounded. Case (1) with $d \geq 2$ is similar (with sufficiently large p_*), and for $d = 1$ one can formally put $p_* = \infty$ by the continuity of the embedding $H_{1/2} \subset C(\overline{\Omega}, \mathbb{C}^n)$.

In case (2) and (3), we define p_γ by (5.3) or (5.2), respectively, and observe that, due to (5.18) or (5.20), respectively, we have the estimate

$$\frac{1}{2} \geq \frac{1}{p_\gamma} + \frac{1}{r}.$$

Hence, by the (generalized) Hölder inequality, $B: L_{p_\gamma}(\Omega, \mathbb{C}^n) \rightarrow L_2(\Omega, \mathbb{C}^n)$ is bounded, and thus also $B: H_{1-\gamma} \rightarrow H$ is bounded by Lemma 5.2. The last assertion follows from Proposition 3.14 and Remark 4.9. \square

If one is interested in stability in H (the case $\alpha = 0$), one should consider Proposition 5.3 part (1) or (2). In the former case, Proposition 5.9(1) is automatically satisfied, and in the latter case one would like to apply Proposition 5.9(2). In the latter case, $\gamma \in [1/2, 1)$ has to satisfy $\gamma_i \leq \gamma \leq \tilde{\gamma}_0$ for $i = 0, 1$ with γ_i from (5.13). Obviously, γ_1 and $\tilde{\gamma}_0$ depend monotonically on r , and $\gamma_1 < \tilde{\gamma}_0$ if r is sufficiently large, and then $\gamma_0 < \tilde{\gamma}_0$ if σ is sufficiently small, so that Proposition 5.3(2) and Proposition 5.9(2) apply simultaneously for all γ from some proper interval (if r is sufficiently large).

If one is interested in stability in V (the case $\alpha = 1/2$), one should consider Proposition 5.3(3). In this case, the hypothesis of Proposition 5.9(1) means an additional requirement for r . The purpose of Proposition 5.9(3) is to relax this requirement. However, it is not immediately clear whether this relaxed requirement applies in the situation of Proposition 5.3(3), since then $\gamma \in [0, 1/2]$ needs to satisfy $\gamma_i \leq \gamma \leq \tilde{\gamma}_p$ for $i = 0, 1$ with γ_i from (5.16). Although γ_1 and $\tilde{\gamma}_p$ depend monotonically on r and satisfy $\gamma_1 < \tilde{\gamma}_p$ if r is sufficiently large, one cannot choose r arbitrarily large in view of (5.19): Otherwise already the additional requirement of Proposition 5.9(1) is satisfied. In fact, the following observation may be somewhat discouraging at a first glance.

Remark 5.10. If (5.19) holds, then the term $\tilde{\gamma}_p$ in (5.20) is strictly increasing with respect to $p \geq \frac{2r}{r-2}$. In particular,

$$\tilde{\gamma}_\infty := \sup_{p \in [\frac{2r}{r-2}, \infty)} \tilde{\gamma}_p = \lim_{p \rightarrow \infty} \tilde{\gamma}_p = \frac{r-2}{4r} p_*.$$

Thus, even if we know that $D(A) \subseteq L_p(\Omega, \mathbb{C}^n)$ for every $p \in (1, \infty)$, we still have $\gamma < \tilde{\gamma}_\infty$, and the latter can be arbitrarily small if r is sufficiently close to 2.

Nevertheless we will show in the following remark that Proposition 5.3(3) and Proposition 5.9(3) apply simultaneously with the same γ provided that r is not “too” small and σ is not “too” large.

Remark 5.11. Suppose that Sobolev’s embedding theorem holds in the sense described earlier and, moreover, that we have a continuous embedding $D(A) \subseteq L_p(\Omega, \mathbb{C}^n)$ with $p = \frac{2d}{d-4}$ in case $d \geq 5$ and any $p \in (p_*, \infty)$ in case $d \leq 4$. For instance, by standard Sobolev embedding theorems (see [20, Theorem 1.4.5]), this is the case if $D(A) \subseteq W^{2,2}(\Omega, \mathbb{C}^n)$. Proposition 5.9(3) applies with

$$\begin{cases} r \in [\frac{d}{2}, d], \gamma \leq \tilde{\gamma}_{2d/(d-4)} = 1 - \frac{d}{2r} & \text{if } d \geq 5, \\ r \in (2, d), \gamma < \tilde{\gamma}_\infty = \frac{r-2}{4r} p_* & \text{if } d = 3, 4. \end{cases}$$

In view of (5.17) it follows that if

$$\begin{cases} r \in [\frac{2}{3}d, d) \text{ and } \sigma \leq \frac{(d+4)r-2d}{(d-2)r} & \text{if } d \geq 5, \\ r \in (\frac{d^2}{2d-2}, d) \text{ and } \sigma < \frac{d^2r-4d}{(d-2)^2r} & \text{if } d = 3, 4, \end{cases}$$

then Proposition 5.3(3) applies with

$$\begin{cases} \max\{\gamma_0, \gamma_1\} \leq \tilde{\gamma}_{2d/(d-4)} & \text{if } d \geq 5, \\ \max\{\gamma_0, \gamma_1\} < \tilde{\gamma}_\infty & \text{if } d = 3, 4. \end{cases}$$

Hence, in these cases there exists $\gamma \in [0, 1/2)$ for which Proposition 5.3(3) and Proposition 5.9(3) apply simultaneously.

A result similar to Proposition 5.3 holds for a Lipschitz condition. We assume that $\tilde{f}: [0, \infty) \times \Omega \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ is single-valued. Let $\tilde{p} \geq 1$, $\sigma > 0$, and $\gamma \in [0, 1/2]$. We define q_γ by (5.1). We assume that for each $t_0 \in [0, \infty)$ there are $L_{t_0} \geq 0$, $\sigma_{t_0} > 0$, and a neighborhood $I \subseteq [0, \infty)$ of t_0 such that for each $t \in I$ there are measurable $a_t, b_t: \Omega \rightarrow [0, \infty)$ with

$$\int_{\Omega} a_t(x)^{\tilde{p}} dx \leq 1 \text{ and } \int_{\Omega} b_t(x)^{q_\gamma} dx \leq 1$$

such that for almost all $x \in \Omega$ the uniform (for all $u, v \in \mathbb{C}^n$) estimate

$$|\tilde{f}(t, x, u) - \tilde{f}(t, x, v)| \leq L_{t_0} \cdot (a_t(x) + |u| + |v|)^{\sigma-1} |u - v| \quad (5.21)$$

holds and such that for each $t, s \in I$ we have for almost all $x \in \Omega$ the uniform (for all $u \in \mathbb{C}^n$) estimate

$$|\tilde{f}(t, x, u) - \tilde{f}(s, x, u)| \leq L_{t_0} (b_t(x) + b_s(x) + |u|^\sigma) |t - s|^{\sigma_{t_0}}. \quad (5.22)$$

Finally, we assume that

$$\tilde{f}(t, \cdot, u) \text{ is measurable for all } (t, u) \in [0, \infty) \times \mathbb{C}^n, \text{ and } \tilde{f}(0, \cdot, 0) \in L_{q_\gamma}(\Omega, \mathbb{C}^n). \quad (5.23)$$

Proposition 5.12. *Let A be a Kato operator, and assume (5.9). Assume one of the following:*

- (1) *Let $\alpha = 0$ and $\gamma \in [1/2, 1)$. Suppose that conditions (5.21), (5.22), and (5.23) hold with $\tilde{p} = 2$ and with σ from (5.11).*
- (2) *Let $\alpha = 0$, and assume that the embedding $D(A^*) \subseteq L_p(\Omega, \mathbb{C}^n)$ is continuous for some $p \in (p_*, \infty)$. Let σ satisfy (5.12), and thus γ_0 from (5.13) satisfies $\gamma_0 < 1$. Let $\gamma \in [\max\{\gamma_0, 0\}, 1)$, and suppose that conditions (5.21), (5.22), and (5.23) hold with $\tilde{p} = 2$.*
- (3) *Let $\alpha = 1/2$. Let σ satisfy (5.15), and thus γ_0 from (5.16) satisfies $\gamma_0 < 1/2$. Let $\gamma \in [\max\{\gamma_0, 0\}, 1/2)$, and suppose that conditions (5.21), (5.22), and (5.23) hold with $\tilde{p} = p_*$.*

Then f maps $[0, \infty) \times H_\alpha$ into $H_{-\gamma}$ and satisfies a right local Hölder-Lipschitz condition (3.3) and is left-locally bounded into $H_{-\gamma}$.

Proof. We use the notation of the proof of Proposition 5.3. Note that (5.23) implies in view of (5.22) by a straightforward estimate that $f(t, 0) \in V_\gamma$ for every $t > 0$. From [16, Appendix] we obtain together with (5.21) that for each $t \in I$ the function $f(t, \cdot)$ maps U into V_γ and satisfies a Lipschitz condition on every bounded set $M \subseteq U$ with Lipschitz constant being independent of $t \in I$. Using (5.22), we find by a straightforward estimate that

$$\|f(t, u) - f(s, u)\|_{V_\gamma} \leq C_{M, t_0} |t - s|^{\sigma_{t_0}} \quad \text{for all } t, s \in I, u \in M,$$

where C_{M,t_0} is independent of $t, s \in I$ and $u \in M$. Combining both assertions and the triangle inequality, we obtain that $f: [0, \infty) \times U \rightarrow V_\gamma$ satisfies a right Hölder-Lipschitz condition and is left-locally bounded into V_γ . Since we have bounded embeddings $H_\alpha \subseteq U$ and $V_\gamma \subseteq (H_\gamma^*)' \cong H_{-\gamma}$ by (4.7), the assertion follows. \square

Remark 5.13. If $\alpha = 0$ and $\gamma \in [0, 1)$ is such that the embedding $H_\gamma \subseteq L_\infty(\Omega, \mathbb{C}^n)$ is continuous, then the conclusion of Proposition 5.12 is also valid (with the same proof and $q_\gamma = 1$, cf. Remark 5.4).

5.2. Semilinear Parabolic PDEs. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Let $\Gamma_D, \Gamma_N \subseteq \partial\Omega$ be disjoint and measurable (with respect to the $(d-1)$ -dimensional Hausdorff measure) and such that

$$(\partial\Omega) \setminus (\Gamma_D \cup \Gamma_N) \text{ is a null set.} \quad (5.24)$$

It is explicitly admissible that $\Gamma_D = \emptyset$ or $\Gamma_N = \emptyset$. Given $a_{j,k}, b_j \in L_\infty(\Omega, \mathbb{C}^{n \times n})$ ($j, k = 1, \dots, d$) and $\tilde{f}: [0, \infty) \times \Omega \times \mathbb{C}^n \rightarrow 2^{\mathbb{C}^n}$, we consider the semilinear PDE

$$\frac{\partial u}{\partial t} + Pu = \tilde{f}_0(t, x, u) \quad \text{on } \Omega, \quad (5.25)$$

where

$$Pw := - \sum_{j,k=1}^d \frac{\partial}{\partial x_j} \left(a_{j,k}(x) \frac{\partial w(x)}{\partial x_k} \right) + \sum_{j=1}^d b_j(x) \frac{\partial w(x)}{\partial x_j}.$$

We impose the mixed boundary condition

$$\begin{cases} u = 0 & \text{on } \Gamma_D, \\ \sum_{j,k=1}^d \nu_j a_{j,k} \frac{\partial u}{\partial x_k} = 0 & \text{on } \Gamma_N, \end{cases} \quad (5.26)$$

where $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ denotes the outer normal at $x \in \partial\Omega$.

We put $H := L_2(\Omega, \mathbb{C}^n)$ and

$$V := \{u \in W^{1,2}(\Omega, \mathbb{C}^n) : u|_{\Gamma_D} = 0 \text{ in the sense of traces}\},$$

equipping V with the norm of $W^{1,2}(\Omega, \mathbb{C}^n)$.

Our main assumption is as follows.

(C): Gårding's inequality holds, that is, there are $c, \tilde{c} > 0$ with

$$\operatorname{Re} \sum_{j,k=1}^d \int_{\Omega} \left(a_{j,k}(x) \frac{\partial u(x)}{\partial x_k} \right) \cdot \overline{\frac{\partial u(x)}{\partial x_j}} dx \geq c \|\nabla u\|_{L_2(\Omega, \mathbb{C}^{dn})}^2 - \tilde{c} \|u\|_{L_2(\Omega, \mathbb{C}^n)}^2 \quad (5.27)$$

for all $u \in V$. Moreover, at least one of the following holds:

- (1) $a_{j,k}(x) = (a_{k,j}(x))^*$ for almost all $x \in \Omega$ and all $j, k = 1, \dots, d$.
- (2) Gårding's inequality (5.27) holds even with $\tilde{c} = 0$. Moreover, Γ_D satisfies the geometric hypotheses described in [9, Assumption 9.1].
- (3) The matrices $\operatorname{Re} \left(\sum_{j,k=1}^d a_{j,k}(x) \xi_j \xi_k \right)$ are positive definite, uniformly with respect

to all $x \in \Omega$ and $\xi \in \mathbb{R}^d$ with $|\xi| = 1$, $a_{j,k} \in C^1(\overline{\Omega}, \mathbb{C}^{n \times n})$, $b_j \in \operatorname{Lip}(\overline{\Omega}, \mathbb{C}^{n \times n})$ for all $j, k = 1, \dots, d$, Γ_D and Γ_N are open in $\partial\Omega$ domains, and the set (5.24) is a Lipschitz manifold of dimension $d-2$.

For a discussion of various algebraic conditions that are sufficient for Gårding's inequality (5.27), we refer the reader to e.g. [2, 21].

By a standard estimate, we obtain from Gårding's inequality (5.27) that the form

$$a(u, v) := \int_{\Omega} \left(\sum_{j,k=1}^d \left(a_{j,k}(x) \frac{\partial u(x)}{\partial x_k} \right) \cdot \frac{\overline{\partial v(x)}}{\partial x_j} + \sum_{j=1}^d \left(b_j(x) \frac{\partial u(x)}{\partial x_j} + Mu(x) \right) \cdot \overline{v(x)} \right) dx$$

satisfies (4.3) if $M \geq 0$ is sufficiently large. Keeping such an M fixed, we now introduce the function

$$\tilde{f}(t, x, u) := \tilde{f}_0(t, x, u) + Mu$$

and define strong (weak, mild) solutions of (5.25), (5.26) as strong (weak, mild) solutions of (2.2) with the superposition operator (5.5). A connection between solutions of (5.25), (5.26) and (2.2) is described in e.g. [27, Theorem 4.4.4].

Theorem 5.14. *Assume that hypothesis (C) holds. Then the operator A associated with a is a Kato operator.*

Moreover, let $(\tilde{f}, \alpha, \gamma)$ satisfy the hypotheses of Proposition 5.3 part (1) or (2) (or (3)), and suppose that there is some $\lambda_0 > 0$ such that every γ -weak eigenvalue λ of $A - B$ satisfies $\operatorname{Re} \lambda \geq \lambda_0$. Then $u_0 = 0$ is asymptotically stable in $H_{\alpha} = L_2(\Omega, \mathbb{C}^n)$ (or $H_{\alpha} = W^{1,2}(\Omega, \mathbb{C}^n)$) in the sense that for every $\varepsilon > 0$ there is $\delta > 0$ such that any γ -mild solution $u \in C([0, \infty), H_{\alpha})$ of (5.25), (5.26) with $\|u(0, \cdot)\|_{H_{\alpha}} \leq \delta$ satisfies $\|u(t, \cdot)\|_{H_{\alpha}} \leq \varepsilon$ for all $t \geq 0$, and $\|u(t, \cdot)\|_{H_{\alpha}} \rightarrow 0$ exponentially fast as $t \rightarrow \infty$.

If in addition \tilde{f} satisfies the hypothesis of Proposition 5.12 part (1) (or (3)), then for every $u_0 \in H_{\alpha}$ there is a unique γ -mild solution $u \in C([0, \infty), H_{\alpha})$ of (5.25), (5.26) with $u(0, \cdot) = u_0$.

Remark 5.15. We emphasize that under the additional assumptions mentioned in Proposition 5.9, it suffices to consider eigenvalues of $A - B$ instead of γ -weak eigenvalues. Note that $A - B$ is actually independent of M (because the terms with M cancel).

Proof. Assume first that $b_1 = \dots = b_d = 0$. Then A is a Kato operator. Indeed, in case (C)(1), this follows from Proposition 4.2 or by Theorem A.5, because a is symmetric. In case (C)(2), this follows from the main result of [9], and in case (C)(3) this follows from the main result of [3] in view of [1].

Since neither the space H_1 nor its topology depends on M or b_j , we obtain from Proposition 4.4 that also the space $H_{1/2} \cong [H, H_1]_{1/2}$ does not depend on M or b_j , and so we obtain from the special case $b_1 = \dots = b_d = 0$ also in the general case that A is a Kato operator.

Note that if the hypothesis of Proposition 5.3(1) is satisfied, then also the hypothesis of Proposition 5.9(1) is satisfied. Hence, the assertion follows from Theorem 3.15. \square

Remark 5.16. In Theorem 5.14, the hypotheses of Proposition 5.3 part (1) or (2) can also be replaced by the hypothesis of Remark 5.4.

Example 5.17. Let $\Omega \subseteq \mathbb{R}^d$ be bounded with a Lipschitz boundary, $\Gamma_D, \Gamma_N \subseteq \partial\Omega$ be measurable with (5.24). Let $f_1, f_2: \mathbb{C}^2 \rightarrow \mathbb{C}$ be continuous with $f_i(0) = 0$, and suppose that there are $L \geq 0$ and $\rho > 0$ with

$$|f_i(u) - f_i(v)| \leq L(1 + |u| + |v|)^{\rho} |u - v| \quad (5.28)$$

for all $u \in \mathbb{C}^2$. Assume that $(b_{i1}, b_{i2}) = f'_i(0)$ exist for $i = 1, 2$, are real, and satisfy the sign conditions

$$b_{11} > 0, \quad b_{11} + b_{22} < 0, \quad b_{11}b_{22} - b_{12}b_{21} > 0.$$

For $d_1, d_2 > 0$, we consider the reaction-diffusion system

$$\frac{\partial u_j}{\partial t} = d_j \Delta u_j + f_j(u_1, u_2) \quad \text{on } \Omega \text{ for } j = 1, 2, \quad (5.29)$$

with mixed boundary conditions (for $u = (u_1, u_2)$)

$$u = 0 \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_N. \quad (5.30)$$

Let $\kappa_k > 0$ ($k = 1, 2, \dots$) denote the nonzero eigenvalues of Δ with boundary conditions (5.30); if Γ_D is a null set, do *not* include the trivial eigenvalue $\kappa_0 = 0$ into this sequence. Suppose (d_1, d_2) lies to the right/under the envelope of the hyperbolas

$$C_k = \{(d_1, d_2) : d_1, d_2 > 0 \text{ and } (\kappa_k d_1 - b_{11})(\kappa_k d_2 - b_{22}) = b_{12}b_{21}\},$$

that is, (d_1, d_2) belongs to

$$\bigcap_{k=1}^{\infty} \left\{ (d_1, d_2) : d_1 \geq \kappa_k^{-1} b_{11} \text{ or } d_2 < \frac{\kappa_k^{-2} b_{12} b_{21}}{d_1 - \kappa_k^{-1} b_{11}} + \frac{b_{22}}{\kappa_k} \right\}, \quad (5.31)$$

see Figure 5.1. Then the following holds in each of the following two cases.

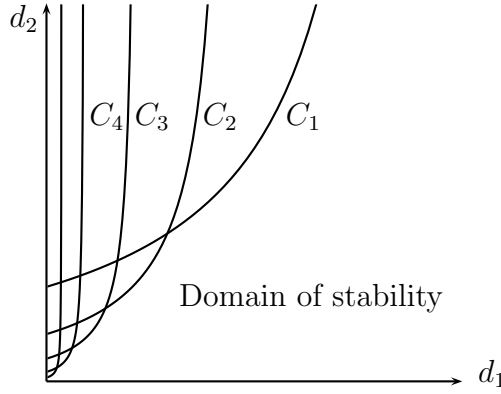


FIGURE 5.1. The hyperbolas C_k

- (1) $H_\alpha = L_2(\Omega, \mathbb{C}^2)$ and one of the following holds:
 - (a) $\gamma \in [1/2, 1)$ and either $d = 1$, $\rho \leq 1$, or $d = 2$, $\rho < 1$, or $d \geq 3$, $\rho \leq 2/d$;
 - (b) $D(A)$ is continuously embedded into $L_p(\Omega, \mathbb{C})$, $\rho < 1 - \frac{2}{p}$, and $\gamma \in (0, 1)$ is sufficiently large;
 - (c) $\rho \leq 1$, $\gamma \in (1/2, 1)$, and H_γ is continuously embedded into $L_\infty(\Omega, \mathbb{C})$;
 - (d) $D(A)$ is continuously embedded into $W^{2,2}(\Omega, \mathbb{C})$, and either $d \leq 3$, $\rho \leq 1$, $\gamma \in (d/4, 1)$, or $d \geq 4$, $\rho < 4/d$, and $\gamma \in (0, 1)$ is sufficiently large.
- (2) $H_\alpha = W^{1,2}(\Omega, \mathbb{C}^2)$ and one of the following holds:
 - (a) $d \leq 2$, $\rho > 0$, $\gamma \in [0, 1/2)$;
 - (b) $d \geq 3$, $\rho < 4/(d-2)$, $\gamma \in [\max\{0, \gamma_0\}, 1/2)$, where γ_0 is defined in (5.16) with $\sigma = \rho + 1$.

For each $\varepsilon > 0$ there is $\delta > 0$ such that for each $u_0 \in H_\alpha$ with $\|u_0\| \leq \delta$ there is a unique γ -mild solution $u \in C([0, \infty), H_\alpha)$ of (5.29), (5.30) with $u(0, \cdot) = u_0$, $\|u(t, \cdot)\|_{H_\alpha} \leq \varepsilon$ for all $t > 0$ and $\|u(t, \cdot)\|_{H_\alpha} \rightarrow 0$ exponentially as $t \rightarrow \infty$.

We first note that (1d) is actually a special case of (1b) and (1c) by the Sobolev embedding theorems and [12, Theorem 1.6.1], respectively. Since f_i is independent of x and t , hypothesis (5.7) follows with $\sigma = \rho + 1$ from (5.28) and from the definition of f'_i . Note

also that the symmetry of A implies $D(A) = D(A^*)$ and $H_\gamma = H_\gamma^*$. The existence and uniqueness assertion follows from Proposition 5.12 or from Remark 5.13 in case (1c). For the stability assertion, we apply Theorem 5.14 or Remark 5.16 in case (1c) with $r = \infty$ and $\sigma = 1 + r$. In view of Proposition 5.9, it thus suffices to verify that there is $\lambda_0 > 0$ such that every eigenvalue λ of $A - B$ satisfies $\operatorname{Re} \lambda \geq \lambda_0$. Under condition (5.31) the latter was verified in [29]. It can be shown by a similar calculation that if $d_i > 0$ violate (5.31) then there is an eigenvalue λ of $A - B$ with $\operatorname{Re} \lambda \leq 0$ ($\lambda = 0$ if $(d_1, d_2) \in \bigcup_{k=1}^\infty C_k$). In this sense, the domain of stability sketched in Figure 5.1 is maximal.

Note that (1d) involves a strictly weaker requirement concerning ρ than (1a) for every $d \geq 2$. The embedding required for (1d) holds in case $\Gamma_D = \emptyset$ or $\Gamma_N = \emptyset$ if $\partial\Omega$ is sufficiently smooth.

The result obtained in [29] concerning Example 5.17 did not cover the case $H_\alpha = L_2(\Omega, \mathbb{C}^2)$. Moreover, even in the case $H_\alpha = W^{1,2}(\Omega, \mathbb{C}^2)$ and $d \geq 3$, the result in [29] essentially needed the more restrictive hypothesis $\rho \leq 2/(d-2)$ which is (almost) by the factor 2 worse than our above requirement for that case.

APPENDIX A. ON THE CHARACTERIZATION OF KATO OPERATORS

As an application of Theorem 4.16, we obtain now a sufficient criterion for Kato operators. In fact, in the following we give a necessary and sufficient condition under which the particular scalar product (4.4) is A -Kato.

Throughout this section, we consider the setting of Section 4. Recall that Proposition 4.2 implies in particular that $A^{-1}: H \rightarrow H$ is bounded. It is well known (see e.g. [15, Theorem III.5.30]) that this implies that also $(A^*)^{-1}: H \rightarrow H$ exists and is bounded and is actually the (bounded) Hilbert-space adjoint $(A^{-1})^*$, i.e.

$$(A^*)^{-1} = (A^{-1})^*. \quad (\text{A.1})$$

Definition A.1. We call A *quasi-symmetric* if there are constants $\alpha > -1$ and $\beta, M \geq 0$ with

$$\operatorname{Re}((A^*)^{-1}(Au + Mu), u) \geq \alpha|u|^2 \quad \text{and} \quad |(A^*)^{-1}Au| \leq \beta|u| \quad \text{for all } u \in D(A). \quad (\text{A.2})$$

If $M \geq 0$ is given, we call A *M -quasi-symmetric* if there are constants $\alpha > -1$, $\beta \geq 0$ with (A.2).

Remark A.2. The larger M is, the less restrictive condition (A.2) becomes. Indeed, (A.1) implies

$$\operatorname{Re}((A^*)^{-1}u, u) = \operatorname{Re}(u, A^{-1}u) = \operatorname{Re}(A(A^{-1}u), A^{-1}u) \geq c\|A^{-1}u\|^2 \geq 0 \quad \text{for all } u \in H. \quad (\text{A.3})$$

Remark A.3. If A is symmetric, then (A.3) implies that A is M -quasi-symmetric with every $M \geq 0$.

Roughly speaking, estimates (A.2) mean indeed that A is quantitatively almost symmetric in the sense that $(A^*)^{-1}A$ does not differ too much from the identity in a quantitative manner, namely that it is “almost” accretive and bounded in H (on the subspace $D(A)$). The restriction $\alpha > -1$ may appear very strange at a first glance, but it is the correct hypothesis for the following result:

Proposition A.4. *For every $M \geq 0$ the following assertions are equivalent.*

- (1) A is M -quasi-symmetric.

(2) There are $\alpha > -1$ and $\tilde{\beta} > 0$ with

$$\operatorname{Re}(Au + Mu, A^{-1}u) \geq \alpha|u|^2 \quad \text{and} \quad |(Au, A^{-1}u)| \leq \tilde{\beta}|u|^2 \quad (\text{A.4})$$

for all $u \in D(A)$.

(3) The formula (4.4) defines an A -Kato scalar product on V .

The relation of the largest possible constants α in (A.2) and (A.4) and c_1 in Proposition 4.10 is given by $2c_1 = 1 + \alpha$.

Proof. For every $u, v \in D(A)$, we obtain from (4.4), the definition of A , and (A.1) that

$$\begin{aligned} 2b_M(u, A^{-1}v) &= a(u, A^{-1}v) + \overline{a(A^{-1}v, u)} + M \cdot (u, A^{-1}v) \\ &= (Au + Mu, A^{-1}v) + \overline{(v, u)} = ((A^*)^{-1}(Au + Mu), v) + (u, v). \end{aligned} \quad (\text{A.5})$$

Hence, if (4.16) or (4.17) hold, then (A.2) or (A.4) hold with $\alpha := 2c_1 - 1$ and some $\beta, \tilde{\beta} > 0$, respectively. Conversely, if (A.2) or (A.4) holds, then (A.5) shows that (4.16) or (4.17) hold with $c_1 := (\alpha + 1)/2$ and some $c_2, c_3 > 0$, respectively. \square

Theorem A.5. *If A is quasi-symmetric, then A is a Kato operator.*

Proof. In view of Proposition A.4, the assertion follows from Theorem 4.16. \square

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PAVEL GUREVICH, FREE UNIVERSITY OF BERLIN, DEPT. OF MATHEMATICS (WE1), ARNIMALLEE 3,
D-14195 BERLIN, GERMANY; PEOPLES' FRIENDSHIP UNIVERSITY OF RUSSIA, MIKLUKHO-MAKLAYA 6,
117198 MOSCOW, RUSSIA

E-mail address: gurevichp@gmail.com

MARTIN VÄTH, MATHEMATICAL INSTITUTE, ACADEMY OF SCIENCES OF THE CZECH REPUBLIC,
ŽITNÁ 25, 115 67 PRAGUE 1, CZECH REPUBLIC

E-mail address: martin@mvath.de